How Good Is a Two-Party Election Game?

Chuang-Chieh Lin[1], Chi-Jen Lu[2], and Po-An Chen[3]

[1] Department of Computer Science and Information Engineering, Tamkang University
No. 151, Yingzhuan Rd., Tamsui Dist., New Taipei City 25137, Taiwan
158778@mail.tku.edu.tw
[2] Institute of Information Science, Academia Sinica
128 Academia Road, Section 2, Nankang, Taipei 11529, Taiwan
cjlu@iis.sinica.edu.tw
[3] Institute of Information Management, National Yang-Ming Chiao-Tung University
1001 University Rd, Hsinchu City 300, Taiwan
poanchen@nctu.edu.tw

Abstract. In this paper, we propose a simple and intuitive model to investigate the efficiency of the two-party election system, especially regarding the nomination process. Each of the two parties has its own candidates, and each of them brings utilities for the people including the supporters and non-supporters. In an election, each party nominates exactly one of its candidates to compete against the other party's. The candidate wins the election with higher odds if he or she brings more utility for all the people. We model such competition as a *two-party election game* such that each party is a player with two or more pure strategies corresponding to its potential candidates, and the payoff of each party is a mixed utility from a selected pair of competing candidates.

By looking into the three models, namely, the linear link, Bradley-Terry, and the softmax models, which differ in how to formulate a candidate's winning odds against the competing candidate, we show that the two-party election game may neither have any pure Nash equilibrium nor a bounded price of anarchy. Nevertheless, by considering the conventional *egoism*, which states that any candidate benefits his/her party's supporters more than any candidate from the competing party does, we prove that the two-party election game in both the linear link model and the softmax model always has pure Nash equilibria, and furthermore, the price of anarchy is constantly bounded.

Keywords: Two-party election game; Nash equilibrium; price of anarchy; egoism

1 Introduction

Most of the literature on voting theory is devoted to studying the voters' behavior on a *micro*-level and equilibria under various election rules. As we know, voters are strategic and have different preferences for the candidates. Their behaviors are sometimes complex and dependent on the voting procedure, which involves the ballot structure and the allocation rule. Due to many factors involved, it is thus difficult for any micro-level model to capture the overall efficiency of a political system consisting of two or multiple competing parties. There are issues and perspectives in the previous works on political competition, listed in the following.

- What kind of voting procedures (e.g., Plurality Voting, Borda Count, Negative Voting, Approval Voting, etc.) should we apply? Moreover, different voting procedures may induce different strategic behaviors of voters.
- What kind of voter preference (e.g., proximity in a metric space) on the policy space is reasonable? Besides, voter preferences may not be identical and may even be dependent.
- Most studies on political competition consider policies or political positions on a "unidimensional" metric space (e.g., a closed interval on the real line), except for very few studies (e.g., López et al. [21] considers political positions as points in a two-dimensional plane). However, multiple policy dimensions are sometimes necessary to capture voter preferences. Then, what is a reasonable dimensionality of the

policy space (or candidate pool)? Besides, such a game of political competition may not have any pure Nash equilibrium when the policy space is multi-dimensional [12].
– In practical cases, the policy space is finite and believed to be countable. For example, it is believed to be in proportion to the number of candidates.

We start our investigation with Duverger's law that suggests plurality voting in favor of the two-party system [13]. There have been studies that formalize Duverger's law. It can be explained either by the strategic behavior of the voters [10, 14, 15, 22, 24] or that of the candidates [5, 6, 23, 28]. (See Section 1.1 for more related work.) The latter is more relevant to our work. It considers models where two selected candidates face a potential entrant, i.e., the third candidate, as a threat, such that the two selected candidates choose policies (i.e., points in the policy space) to preempt successful entry. Under mild assumptions on voter preferences, Dellis [10] explained why a two-party system emerges under plurality voting and other voting procedures permitting truncated ballots. However, it is not explained why and how the two candidates are selected, which is exactly what we are interested in this article.

We observe that modern democracy runs on political parties nominating their "best" candidates to compete in elections by appealing mainly to their supporters. As far as we know, not much has been said about *how the parties form their nomination strategies in terms of candidate selection in an election, and how this process stabilizes and affects the benefits of the people in the end.* Usually, a candidate nominated by a party responds and caters more to the needs of his/her party's supporters and less to the needs of the non-supporters. This subsequently determines different benefits/utilities that a candidate could bring to the supporters and to the non-supporters.[4] Furthermore, by Duverger's law suggesting that the plurality voting favors the two-party system [13], we consider a democracy consisting of two parties. Such examples of two-party or nearly two-party systems exist in many democratic countries such as the United States, the United Kingdom, Taiwan, etc.

Here, we consider a simplified *macro*-model instead. We formalize the political competition between two parties as a *two-player election game*, in which each party corresponds to a player having candidates as its strategies and its payoff is the expected utility for its supporters considering that their selected candidate may win or lose. The odds of winning an election for a candidate nominated by a party over another candidate nominated by the competing party could arguably be related to two factors - the total benefits that he/she brings to the whole society, including the supporters and non-supporters, and those that his/her competitor brings.

Based on the proposed simple and intuitive models, we would like to know how the two-party system benefits people by asking whether a two-party system will stabilize, i.e., it has an equilibrium, and in particular whether the competition between the two parties in such a system always benefits the people optimally. If not, then how good/bad can the selected pair of competing candidates be at equilibrium, compared with the optimal pair? Specifically, we answer the former question by proving the existence of a Nash equilibrium, and analyze the latter in terms of the price of anarchy [18] which measures the efficiency of an equilibrium of a game. By considering the conventional *egoism*, which states that any candidate benefits his/her party's supporters more than any candidate from the competing party does, we prove that the two-party election game in some models always has pure Nash equilibria, and the price of anarchy is constantly bounded. This shows that in some sense the game between two parties in candidate nomination for an election benefits the people with a social welfare at most constantly far from the optimum.

Discussion of Our Results

In this paper, we bypass the issues mentioned in the beginning of introduction by modeling the political competition as a two-player game, which is called a *two-party election game*, and analyze the efficiency of the game on a macro level. Specifically, the two parties are modeled as two strategic players, and each player's strategies correspond to its candidates, which are countably many and finite. Each candidate of a party brings

[4] Note that a voter who obtains the same amount of benefits from any party's candidate can be considered as a supporter of an arbitrary party.

utilities for the supporters of its party and also for the supporters of the competing party. A candidate wins the competition with higher odds if one brings more utility for all the voters. Each player's payoff is the expected utility its supporters get from the two parties. Moreover, we conjecture that the winning probability of a candidate is a function of the total utility it can bring. By considering three kinds of functions modeling the winning probability, we investigate if such a nomination competition between two parties always has a pure Nash equilibrium. Moreover, if the answer is affirmative, we analyze the price of anarchy of the game.

The setting of our model, which is inspired by dueling bandits [1, 29], is simple and intuitive. We prove that the two-party election game may neither have any pure Nash equilibrium nor bounded price of anarchy. However, under the standard assumption of *egoism*, which states that any candidate benefits its supporters more than those from the opposite party, we prove that the two-party election game in both the linear link model and the softmax model always has pure Nash equilibria, and furthermore, the price of anarchy is bounded by a constant. See Section 5 for examples in non-egoistic games. We also show examples of no existence of equilibria for some models with or without egoism in Section 3.1. Our findings suggest that, under the egoism assumption, the two-party system is stable and efficient for the people. Note that we are modeling in this paper the decision making about the candidate choices of a two-party election system, depending on the utility estimates for the supporters and non-supports. This is less about the voters' behavior or how the voters are modeled. We focus on the candidate choice instead of voting itself, which has already been emphasized in most of the voting theory. We position our work in comparison to some related work that may be different from or similar to ours in terms of models, measures and results in the next subsection.

1.1 Related work

Compared with equilibria in other forms of political competition, most of the works on equilibria of a political competition are mainly based on *Spatial Theory of Voting* (e.g., [11, 21, 23, 28], which was initialized by Downs [11] and can be traced back to Hotelling's work [16]). In the Hotelling-Downs model, there are two established parties facing voters with symmetric single-peaked preferences over a unidimensional metric space. Each party chooses a political position (or, a policy) that is as close as possible to the greatest number of voter preferences. The Spatial Theory of Voting states that when the policy (i.e., candidate) space is unidimensional, voter preferences are single-peaked and two parties compete for one position, the parties' strategies would be determined by the median voter's preference. However, for political positions or policies over a multi-dimensional space, pure Nash equilibria may not exist [12].

In the aspect of efficiency measure of equilibria, another notion "distortion" introduced by Procaccia and Rosenschein [25] resembles the notion of price of anarchy, although the latter is used in games of strategic players. The distortion measures the rate of social welfare decrease (or social cost increase) when voting is used. The related work includes the studies which focus on cardinal preferences of voters (e.g., [7, 25]) and those which focuses on metric preferences of voters (e.g., [2, 3, 9]). Caragiannis and Procaccia [7] discussed the distortion when each voter's vote receives an embedding, which maps the preference to the output vote. Cheng et al. [9] restricted their attention to the distribution of voters and candidates and justified that the expected distortion is small when candidates are drawn from the same distribution of voters.

Furthermore, political competition that we consider in this article is in form of a simultaneous one-shot game, but such competition can also be modeled more dynamically as a multi-round process. In a *multi-battle contest* [17], each time a player competes by investing some of her budget or resource in a component battle to collect a value if winning the battle. There are multiple battles for the players to fight, and the budget get consumed over time. The final result of such a contest is determined by the outcomes of all these multiple battles. Examples include R&D races, sports competition, elections, and many more. The final winner in the overall contest is the one who first reaches some amount of accumulated value or has a majority (or plurality) in value. Focusing on the strategies, a player needs to make adequate sequential actions to win the contest against dynamic competition over time from the others [8].

2 Preliminaries

Let A, B be two political parties devoting to an election, such that party A has m candidates $A_1, \dots A_m$ and party B has n candidates $B_1, \dots, B_n$. Each party has to select one of its candidates to participate in the election. Without loss of generality, $m \geq 2, n \geq 2$. Assume that the society consists of the supporters of A and those of B. Let $u_A(A_i)$ and $u_B(A_i)$ denote party A supporters' total utilities when candidate A_i is elected and party B supporters' total utilities when candidate A_i is elected, respectively, for each $i \in [m]$. Likewise, let $u_A(B_j)$ and $u_B(B_j)$ denote for party A supporters' total utilities when candidate B_j is elected and party B supporters' total utilities when candidate B_j is elected, respectively, for each $j \in [n]$. Assume that candidates in each party are sorted according to the utilities for his or her party's supporters. Namely, $u_A(A_1) \geq u_A(A_2) \geq \dots \geq u_A(A_m)$ and $u_B(B_1) \geq u_B(B_2) \geq \dots \geq u_B(B_n)$. To break the symmetry, assume that $u_A(A_1) \geq u_B(B_1)$.[5] We use $u(A_i)$ (resp., $u(B_j)$) to denote the total utilities for the whole society when candidate A_i (resp., B_j) is elected, i.e.,

$$u(A_i) = u_A(A_i) + u_B(A_i)$$

$$(resp., u(B_j) = u_A(B_j) + u_B(B_j)).$$

Assume that the social utilities are bounded. That is, $u(A_i), u(B_j) \in [0, b]$ for some real $b \geq 1$, for each $i \in [m], j \in [n]$.

Parties A and B represent the two players such that A and B have m and n strategies, respectively. A pure strategy of a party is a selected candidate for the election. An important property that we want to preserve in the game is the following: *a party wins the election with higher odds if it selects a candidate with a higher social utility defined above.* The odds of winning then depend on the social utilities brought by the candidates. Hence, we formulate the winning odds, $p_{i,j}$, which stands for the probability of A_i winning over B_j, in a way to preserve the property as below so that we can formally give the payoff for each party in an election game. Also, note that the probability of B_j winning over A_i is $1 - p_{i,j}$, *not* $p_{j,i}$.

– Linear link model [1]:

$$p_{i,j} := \frac{1 + (u(A_i) - u(B_j))/b}{2}.$$

- This is inspired by the exploration method used in the multi-armed bandit problem [20] and the probabilistic comparison used in the dueling bandits problem [1, 29]. The winning odds is then regarded as a *linear* function of the *difference* between the social utilities brought by candidates A_i and B_j.

– Softmax model [20, 27][6]:

$$p_{i,j} := \frac{e^{u(A_i)/b}}{e^{u(A_i)/b} + e^{u(B_j)/b}}.$$

- Softmax function is extensively used in machine learning to normalize the output into a probability distribution. Naturally, we consider the softmax function as a bivariate *nonlinear* rational function of the *social utilities* brought by candidates A_i and B_j.

– Bradley-Terry model [4, 29]:

$$p_{i,j} := \frac{u(A_i)}{u(A_i) + u(B_j)}.$$

- In this model, the winning probability $p_{i,j}$ is a function analogous to softmax, yet it has *linear* dependency on the two candidates' social utilities.

[5] This means that party A is the party with the higher or equal utility from its best candidate A_1 to its own supporters than party B's utility from its best candidate B_1 to its own supporters. They are not exchangeable.

[6] Actually, we consider a constrained softmax model here. The winning probability is based on the Boltzmann distribution with $kT = b$, where k is the Boltzmann constant and T is the temperature of the system

2.1 Two-Party Election Game

Now, we are ready to use the concepts introduced above to define the payoffs of our two-party election game. When the context is clear, we use (i, j) to denote the state of game in which A_i and B_j are selected in the election. The payoff of party A (resp., B) in state (i, j), say $a_{i,j}$ (resp., $b_{i,j}$), is defined as the expected utilities (or, equivalently, the portion of social welfare) that party A supporters (resp., party B supporters), obtain in state (i, j). Namely,

$$\begin{aligned} a_{i,j} &= p_{i,j} u_A(A_i) + (1 - p_{i,j}) u_A(B_j) \\ b_{i,j} &= (1 - p_{i,j}) u_B(B_j) + p_{i,j} u_B(A_i). \end{aligned}$$

The *social welfare* of state (i, j) is $SU_{i,j} = a_{i,j} + b_{i,j}$. We say that a state (i, j) is a *pure Nash equilibrium* (PNE) if $a_{i',j} \le a_{i,j}$ for any $i' \ne i$ and $b_{i,j'} \le b_{i,j}$ for any $j' \ne j$. That is, in state (i, j), neither A nor B wants to deviate from his or her strategy. The *price of anarchy* (PoA) of the game is defined as

$$\frac{SU_{i^*,j^*}}{SU_{\hat{i},\hat{j}}} = \frac{a_{i^*,j^*} + b_{i^*,j^*}}{a_{\hat{i},\hat{j}} + b_{\hat{i},\hat{j}}},$$

where $(i^*, j^*) = \arg\max_{(i,j)\in[m]\times[n]}(a_{i,j} + b_{i,j})$ is the *optimal state*, which has the best (i.e., highest) social welfare among all possible states, and $(\hat{i}, \hat{j}) = \arg\min_{\substack{(i,j)\in[m]\times[n] \\ (i,j) \text{ is a PNE}}} (a_{i,j} + b_{i,j})$ is the PNE with the worst (i.e., lowest) social welfare.

In an election, probabilistically (instead of deterministically) nominating a candidate or imaging repeated nominations of candidates is almost infeasible in the reality. Thus, adopting PNE as the equilibrium concept best reflects the situation of an election. Nonetheless, it is theoretically natural for one to consider mixed Nash equilibria or other more general notions of equilibria as solution concepts where the existence of equilibria is always guaranteed. It may take other analysis frameworks such as [19, 26] to study their corresponding price-of-anarchy bounds.

The following properties will be needed throughout the article.

Definition 1. *We say that the two-party election game is* egoistic *if $u_A(A_i) > u_A(B_j)$ and $u_B(B_j) > u_B(A_i)$ for all $i \in [m], j \in [n]$.*

This guarantees that *any candidate benefits its supporters more than those from the competing party*, which is reasonable and consistent with the real world.

Definition 2. *For party $X \in \{A, B\}$, we say that strategy i* weakly surpasses *i' if $i < i'$ and $u(X_i) \ge u(X_{i'})$. We say that strategy i* surpasses *i' if i weakly surpasses i' and $u(X_i) > u(X_{i'})$.*

Remark 1. For all the three models, $p_{i,j} \ge p_{i',j}$ if and only if $u(A_i) \ge u(A_{i'})$.

This means that given party B's choice of candidate j, if the total utilities when party A's candidate i is elected are greater than or equal to those when party A's candidate i' is elected, then the chance of candidate i winning over candidate j is greater than or equal to that of candidate i' winning over candidate j, and vice versa.

Lemma 1. *If strategy 1 weakly surpasses each $i \in [n] \setminus \{1\}$ in party A, then $(1, j^\#)$ is a PNE of the egoistic two-party election game for $j^\# = \arg\max_{j\in[m]} b_{1,j}$. Similarly, if strategy 1 weakly surpasses each $j \in [m] \setminus \{1\}$ in party B, then $(i^\#, 1)$ is a PNE for $i^\# = \arg\max_{i\in[n]} a_{i,1}$.*

Proof. Suppose that strategy 1 weakly surpasses each $i \in [n] \setminus \{1\}$ in party A and B chooses strategy j. Clearly, we have $p_{1,j} \ge p_{i,j}$ for any $i \ne 1$. Since

$$\begin{aligned} a_{1,j} - a_{i,j} &= p_{1,j} u_A(A_1) + (1 - p_{1,j}) u_A(B_j) \\ &\quad - (p_{i,j} u_A(A_i) + (1 - p_{i,j}) u_A(B_j)) \\ &\ge u_A(A_1)(p_{1,j} - p_{i,j}) + (p_{i,j} - p_{1,j}) u_A(B_j) \\ &= (p_{1,j} - p_{i,j})(u_A(A_1) - u_A(B_j)) \\ &\ge 0 \end{aligned}$$

in which the last inequality follows from the egoistic property, A always wants to choose strategy 1. The best response of B is then choose $j^{\#} = \arg\max_{j\in[m]} b_{1,j}$, and hence $(1, j^{\#})$ is a PNE. The other case can be analogously proved. □

Namely, if strategy 1 in A (weakly) surpasses each $i \in [n] \setminus \{1\}$ and strategy 1 in B (weakly) surpasses each $j \in [m]$, then $(1, 1)$ is a (weakly) dominant-strategy solution.

Lemma 2. *Let $r, s > 0$ be two positive real numbers. Then, for any $d > 0$, $r/s > (r + d)/(s + d)$ if $r > s$ and $r/s < (r + d)/(s + d)$ if $r < s$.*

Proof. Let $r, s > 0$ be two positive real numbers. Then $r/s - (r + d)/(s + d) = d(r - s)/(s^2 + sd)$. So $r/s - (r + d)/(s + d) > 0$ if $r > s$ and $r/s - (r + d)/(s + d) < 0$ if $r < s$. □

3 Equilibrium Analysis

We first show that the two party in the Bradley-Terry model, a PNE does not always exist no matter whether egoism exists or not. Throughout the rest of Section 3, the existence of PNE in *egoistic games* is studied. We then start with the case of two candidates per party, and with the help of Lemma 1 prove existence of PNE in both the linear link model and the softmax model, also using critical lemmas dealing with the complemented condition in Lemma 1. We eventually reuse these critical lemmas to show existence of PNE for the case of more than two candidates per party.

3.1 No Existence Guarantee of PNE in the Bradley-Terry Model

We claim that the two party election game in the Bradley-Terry model, a PNE does not always exist. As the upper instance illustrated in Table 1, $m = n = 2$, $u_A(A_1) = 91$, $u_B(A_1) = 0$, $u_A(A_2) = 90$, $u_B(A_2) = 8$, $u_B(B_1) = 11$, $u_A(B_1) = 1$, $u_B(B_2) = 10$, $u_A(B_2) = 20$. We have $p_{1,1} = 91/(91 + 12) \approx 0.88$, $p_{1,2} = 91/(91 + 30) \approx 0.76$, $p_{2,1} = 98/(98 + 12) \approx 0.89$, and $p_{2,2} = 98/(98 + 30) \approx 0.77$. Hence, we obtain the first matrix in Table 1. Furthermore, as the example is egoistic, it also implies that the game in the Bradley-Terry model may not have a PNE even with egoism guarantee. The second instance in Table 1 gives a non-egoistic example of no PNE in the Bradley-Terry model.

A		B	
$u_A(A_i)$	$u_B(A_i)$	$u_B(B_j)$	$u_A(B_j)$
91	0	11	1
90	8	10	20

A		B	
$u_A(A_i)$	$u_B(A_i)$	$u_B(B_j)$	$u_A(B_j)$
44	10	37	17
39	55	10	5

	B_1	B_2
A_1	$a_{1,1}$, $b_{1,1}$	$a_{1,2}$, $b_{1,2}$
A_2	$a_{2,1}$, $b_{2,1}$	$a_{2,2}$, $b_{2,2}$

$\approx$

	B_1	B_2
A_1	80.51, 1.28	73.84, 2.17
A_2	80.29, 8.32	74.02, 8.23

,

	B_1	B_2
A_1	30.50, 23.50	35.52, 10.00
A_2	30.97, 48.43	34.32, 48.81

Table 1. Two examples of No PNE in the Bradley-Terry model ($m = n = 2, b = 100$). Left one is egoistic while the right one is not.

3.2 Two Candidates per Party

There are exactly two scenarios for the two-party election game with $m = n = 2$ to have no pure Nash equilibrium, as illustrated in Fig. 1. The arrows show the deviations. For example, (D_1) stands for A's unilateral deviation from strategy 1 to 2 given B staying at strategy 1 and (D_2) stands for B's unilateral deviation from strategy 1 to 2 given A staying at strategy 2, while (D_3) stands for A's unilateral deviation from strategy 2 to strategy 1 given B staying at strategy 2 and (D_4) stands for B's unilateral deviation from strategy 2 to strategy 1 given A staying at strategy 1.

That is, if the game has no PNE, then at state $(1,1)$, either A or B deviates from strategy 1 to 2. For the former case, since no PNE exists, B wants to deviate unilaterally from strategy 1 to 2 at state $(2,1)$, then the game reaches state $(1,2)$, in which A wants to deviate unilaterally from strategy 2 to 1. Finally, the entries in the payoff matrix as well as the deviation arrows form a cycle as the left scenario of Fig. 1 shows. Likewise, the latter case corresponds to the right scenario of the Fig. 1.

eps/deviation_NoPNE-eps-converted-to.pdf

Fig. 1. The two scenarios of having no PNE for $m = n = 2$.

Let $\Delta(D_i)$ (resp., $\Delta(D'_i)$) denote the gain of payoff by the unilateral deviation D_i (resp., D'_i) for $i \in \{1,2,3,4\}$. Then, we have

$$\begin{aligned}
\Delta(D_1) = -\Delta(D'_1) &= a_{2,1} - a_{1,1} \\
&= p_{2,1}u_A(A_2) + (1-p_{2,1})u_A(B_1) \\
&\quad -(p_{1,1}u_A(A_1) + (1-p_{1,1})u_A(B_1)) \\
&= -p_{1,1}(u_A(A_1) - u_A(A_2)) \\
&\quad +(p_{2,1} - p_{1,1})(u_A(A_2) - u_A(B_1)). \\
\Delta(D_2) = -\Delta(D'_2) &= b_{2,2} - b_{2,1} \\
&= (1-p_{2,2})u_B(B_2) + p_{2,2}u_B(A_2) \\
&\quad -((1-p_{2,1})u_B(B_1) + p_{2,1}u_B(A_2)) \\
&= -(1-p_{2,1})(u_B(B_1) - u_B(B_2)) \\
&\quad +(p_{2,1} - p_{2,2})(u_B(B_2) - u_B(A_2)).
\end{aligned}$$

Similarly, we derive

$$\begin{aligned}
\Delta(D_3) = -\Delta(D'_3) &= a_{1,2} - a_{2,2} \\
&= p_{1,2}u_A(A_1) + (1-p_{1,2})u_A(B_2) \\
&\quad -(p_{2,2}u_A(A_2) + (1-p_{2,2})u_A(B_2)) \\
&= p_{1,2}(u_A(A_1) - u_A(A_2)) \\
&\quad +(p_{1,2} - p_{2,2})(u_A(A_2) - u_A(B_2)). \\
\Delta(D_4) = -\Delta(D'_4) &= b_{1,1} - b_{1,2} \\
&= (1-p_{1,1})u_B(B_1) + p_{1,1}u_B(A_1) \\
&\quad -((1-p_{1,2})u_B(B_2) + p_{1,2}u_B(A_1)) \\
&= (1-p_{1,1})(u_B(B_1) - u_B(B_2)) \\
&\quad +(p_{1,2} - p_{1,1})(u_B(B_2) - u_B(A_1)).
\end{aligned}$$

In order to check if the two-party election game with $m = n = 2$ always has a PNE, Lemma 1 suggests us to focus on the case that $u(A_2) > u(A_1)$ and $u(B_2) > u(B_1)$ since when either $u(A_1) \geq u(A_2)$ or $u(B_1) \geq u(B_2)$ a PNE always exists.

Linear Link Model In this subsection, we study the linear link model. We show that the two-party election game always has a PNE in this model. As previously discussed in this section, we assume that $u(A_2) > u(A_1)$ and $u(B_2) > u(B_1)$. The following lemma tells us at least one of the deviations D_2 and D_4 fails to happen, and analogously, at least one of deviations D_1' and D_3' fails to happen.

Lemma 3. *Consider the two-party election game in the linear link model. Suppose that $u(A_2) > u(A_1)$, then $\Delta(D_4) < 0$ (resp., $\Delta(D_2) < 0$) if $\Delta(D_2) > 0$ (resp., $\Delta(D_4) > 0$). Likewise, suppose that $u(B_2) > u(B_1)$, then $\Delta(D_3') < 0$ (resp., $\Delta(D_1') < 0$) if $\Delta(D_1') > 0$ (resp., $\Delta(D_3') > 0$).*

Proof. Suppose we have $u(A_2) > u(A_1)$. To ease the notation, let $\hat{p} = 1 - p_{2,1}$, $p' = 1 - p_{1,1}$, and $\delta = p_{2,1} - p_{2,2} = p_{1,1} - p_{1,2} = (u(B_2) - u(B_1))/2b$.

To prove the lemma by contradiction, we assume and focus that $\Delta(D_2) > 0$ yet $\Delta(D_4) \geq 0$, as the other case that $\Delta(D_4) > 0$ yet $\Delta(D_2) \geq 0$ can be proved in the same way. First, from the definitions of D_4 and D_2, we have

$$\begin{cases} p'(u_B(B_1) - u_B(B_2)) \geq \delta \cdot (u_B(B_2) - u_B(A_1)), \\ \hat{p}(u_B(B_1) - u_B(B_2)) < \delta \cdot (u_B(B_2) - u_B(A_2)). \end{cases}$$

We claim that $\hat{p}, p' > 0$. Recall that $p_{1,1} \geq p_{2,1}$ which implies that $\hat{p} \geq p'$, then $p' = 0$ must be true if one of $\hat{p}$ and p' is 0. Substituting $p' = 0$ in the first inequality above we have $0 \geq \delta \cdot (u_B(B_2) - u_B(A_1))$, which contradicts that $u_B(B_2) > u_B(A_1)$ by egoism assumption. Dividing the inequalities by p' and $\hat{p}$ respectively, we have

$$\begin{cases} u_B(B_1) - u_B(B_2) \geq \delta \cdot (u_B(B_2) - u_B(A_1))/p', \\ u_B(B_1) - u_B(B_2) < \delta \cdot (u_B(B_2) - u_B(A_2))/\hat{p}. \end{cases}$$

Note that $u_B(B_2) > u_B(A_2)$ and $u_B(B_2) > u_B(A_1)$ due to egoism, so $\delta \leq 0$ will make the second inequality fail to hold. Hence we proceed with $\delta > 0$ and compare the right-hand sides of the above inequalities:

$$\begin{aligned} &\frac{(u_B(B_2) - u_B(A_1))/p'}{(u_B(B_2) - u_B(A_2))/\hat{p}} \\ &= \frac{u_B(B_2) - u_B(A_1)}{u_B(B_2) - u_B(A_2)} \cdot \frac{1 + (u(B_1) - u(A_2))/b}{1 + (u(B_1) - u(A_1))/b} \\ &= \frac{u_B(B_2) - u_B(A_1)}{u_B(B_2) - u_B(A_2)} \cdot \frac{b + (u(B_1) - u(A_2))}{b + (u(B_1) - u(A_1))}. \end{aligned} \tag{1}$$

Note that $u_B(A_2) > u_B(A_1)$ since $u(A_2) > u(A_1)$ and $u_A(A_1) \geq u_A(A_2)$. Then we have

$$\begin{aligned} &\frac{u_B(B_2) - u_B(A_1)}{u_B(B_2) - u_B(A_2)} \\ &\geq \frac{u_B(B_2) - u_B(A_1) + (u(B_1) - u_B(B_2))}{u_B(B_2) - u_B(A_2) + (u(B_1) - u_B(B_2))} \quad \text{(by Lemma 2)} \\ &= \frac{u(B_1) - u_B(A_1)}{u(B_1) - u_B(A_2)} \\ &\geq \frac{u(B_1) - u_B(A_1) + (b - u_A(A_1))}{u(B_1) - u_B(A_2) + (b - u_A(A_1))} \quad \text{(by Lemma 2)} \\ &\geq \frac{u(B_1) - u_B(A_1) + (b - u_A(A_1))}{u(B_1) - u_B(A_2) + (b - u_A(A_2))} \quad (\text{since } u_A(A_1) \geq u_A(A_2)) \\ &= \frac{u(B_1) - u(A_1) + b}{u(B_1) - u(A_2) + b}, \end{aligned}$$

Finally, together with equation (1), we derive that

$$\begin{aligned}&\frac{(u_B(B_2)-u_B(A_1))/p'}{(u_B(B_2)-u_B(A_2))/\hat{p}}\\ \geq\ &\frac{u(B_1)-u(A_1)+b}{u(B_1)-u(A_2)+b}\cdot\frac{b+(u(B_1)-u(A_2))}{b+(u(B_1)-u(A_1))}=1,\end{aligned}$$

which implies that $u_B(B_1)-u_B(B_2)>u_B(B_1)-u_B(B_2)$, hence a contradiction occurs. For the second part of the lemma that $u(B_2)>u(B_1)$, the proof can be similarly derived. □

Theorem 1 holds by Lemma 1 and 3.

Theorem 1. *In the linear link model with $m=n=2$, the two-party election game always has a PNE.*

Softmax Model In this subsection, we show that the two-party election game always has a PNE of in the softmax model.

Lemma 4. *Consider the two-party election game in the softmax model. Suppose that $u(A_2)>u(A_1)$, then $\Delta(D_4)<0$ (resp., $\Delta(D_2)<0$) if $\Delta(D_2)>0$ (resp., $\Delta(D_4)>0$). Likewise, suppose that $u(B_2)>u(B_1)$, then $\Delta(D_3')<0$ (resp., $\Delta(D_1')<0$) if $\Delta(D_1')>0$ (resp., $\Delta(D_3')>0$).*

Proof. Suppose that $u(B_2)>u(B_1)$. To ease the notation, let

$$\begin{aligned}&q=p_{1,1}=\frac{e^{u(A_1)/b}}{e^{u(A_1)/b}+e^{u(B_1)/b}},\ q'=p_{2,1}=\frac{e^{u(A_2)/b}}{e^{u(A_2)/b}+e^{u(B_1)/b}},\\ &\delta=q'-q.\\ &\hat{q}=p_{1,2}=\frac{e^{u(A_1)/b}}{e^{u(A_1)/b}+e^{u(B_2)/b}},\ \hat{q}'=p_{2,2}=\frac{e^{u(A_2)/b}}{e^{u(A_2)/b}+e^{u(B_2)/b}},\\ &\delta'=\hat{q}'-\hat{q}.\end{aligned}$$

To prove the lemma by contradiction, similarly to the argument in the proof of Lemma 3, we focus on the case that $\Delta(D_3')>0$ yet $\Delta(D_1')\geq 0$. The proof for the other case that $\Delta(D_1')>0$ yet $\Delta(D_3')\geq 0$ is basically the same.

By definition, $q,q',\hat{q},\hat{q'}>0$.

First, as $q,\hat{q}>0$, $\Delta(D_3')>0$ and $\Delta(D_1')\geq 0$ implies that

$$\begin{aligned}&u_A(A_1)-u_A(A_2)\geq\delta\cdot\frac{u_A(A_2)-u_A(B_1)}{q},\ \text{and}\\ &u_A(A_1)-u_A(A_2)<\delta'\cdot\frac{u_A(A_2)-u_A(B_2)}{\hat{q}}.\end{aligned}$$

By the same argument exploited in the proof of Lemma 3, it follows that $\delta'>0$ otherwise the second inequality fails. This implies $u(A_2)>u(A_1)$ so that $\delta>0$. Then we have

$$\begin{cases}u_A(A_1)-u_A(A_2)\\ \geq\frac{e^{u(B_1)/b}(e^{u(A_2)/b}-e^{u(A_1)/b})}{e^{(u(A_1)+u(A_2))/b}+e^{(u(A_1)+u(B_1))/b}}\cdot(u_A(A_2)-u_A(B_1))\triangleq(*)\\ u_A(A_1)-u_A(A_2)\\ <\frac{e^{u(B_2)/b}(e^{u(A_2)/b}-e^{u(A_1)/b})}{e^{(u(A_1)+u(A_2))/b}+e^{(u(A_1)+u(B_2))/b}}\cdot(u_A(A_2)-u_A(B_2))\triangleq(**)\end{cases}$$

Then, we compare the right-hand sides of the above two inequalities as below.

$$\begin{aligned}\frac{(*)}{(**)}&=\frac{e^{u(B_1)/b}}{e^{u(B_2)/b}}\cdot\frac{e^{u(A_2)/b}+e^{u(B_2)/b}}{e^{u(A_2)/b}+e^{u(B_1)/b}}\cdot\frac{u_A(A_2)-u_A(B_1)}{u_A(A_2)-u_A(B_2)}\\ &\geq e^{(u(B_1)-u(B_2))/b}\cdot\frac{u_A(A_2)-u_A(B_1)}{u_A(A_2)-u_A(B_2)}.\end{aligned}$$

Let $c = u_A(A_2) - u_A(B_2)$ and $\rho = u(B_2) - u(B_1) > 0$. Then, $u_A(A_2) - u_A(B_1) = c + u_A(B_2) - u_A(B_1) \geq c + (u(B_2) - u(B_1)) = c + \rho$. Note that $e^{(u(B_1)-u(B_2))/b} \geq 1 + (u(B_1) - u(B_2))/b = 1 - \rho/b$. Also, $(u_A(A_2) - u_A(B_1))/(u_A(A_2) - u_A(B_2)) \geq (c + \rho)/c = 1 + \rho/c$. Therefore,

$$\begin{aligned}\frac{(*)}{(**)} &\geq \left(1 - \frac{\rho}{b}\right) \cdot \left(1 + \frac{\rho}{c}\right) \\ &= 1 + \frac{\rho}{c} - \frac{\rho}{b} - \frac{\rho^2}{bc} \\ &= 1 + \rho \cdot \frac{b - c - \rho}{bc} \\ &\geq 1.\end{aligned}$$

where the last inequality follows because $b - c - \rho = b - u_A(A_2) + u_A(B_2) - u_B(B_2) - u_A(B_2) + u_B(B_1) + u_A(B_1) = (b - u_A(A_2)) + (u_B(B_1) - u_B(B_2)) + u_A(B_1) \geq 0$. Finally, we obtain a contradiction. The other part of the lemma that $u(B_2) > u(B_1)$ can be similarly proved. □

By Lemma 1 and 4, we obtain Theorem 2 as follows.

Theorem 2. *In the softmax model with $m = n = 2$, the two-party election game always has a PNE.*

3.3 More than Two Candidates per Party

In this section, we show existence of PNE for a generalized case in which party A has $m \geq 2$ candidates and party B has $n \geq 2$ candidates. Since the game in the Bradley-Terry model may not have a PNE, we focus on the linear link model and the softmax model.

The two-party election game then has mn possible states (i.e., $\{(i, j)\}_{i \in [m], j \in [n]}$). The states are bijectively mapped to the entries of the payoff matrix. Regard each entry of the payoff matrix as a node and a unilateral deviation with positive gain of payoff as an arc, we obtain a directed graph and we call it the *state graph*. A *best-response walk* is a walk on the state graph such that each arc of the walk is a best-response unilateral deviation. Since the number of states is finite, any best-response walk on the state graph must contain a loop if the game has no PNE. We then derive Theorem 3 based on this observation.

eps/generalNE-eps-converted-to.pdf

Fig. 2. Illustration for the proof of Theorem 3.

Theorem 3. *The two-party election game with $m \geq 2$ and $n \geq 2$ always has a PNE both in the linear link model and the softmax model.*

Proof. Assume, for contradiction, that the game has no PNE. Then, a best-response walk starting from any node contains a loop. Let $L = (i, j) \to (i', j) \to (i', j'') \to \cdots \to (i, j') \to (i, j)$ be such a loop, in which i is the smallest indexed party A candidate among all nodes (states) of L (See Fig. 2 for the illustration). Let α_1 and α_2 denote the unilateral deviations $(i, j'') \to (i, j)$ and $(i', j) \to (i', j'')$ respectively. Since α_2 is on L, we have $\Delta(\alpha_2) > 0$. Since $(i,j') \to (i,j)$ is the best response of B when A selects i, it must be that $b_{i,j} \geq b_{i,j''}$. Thus, if A selects i and B selects j'', then we must have $\Delta(\alpha_1) \geq 0$.

Moreover, since $u(A_i) \geq u(A_{i'})$ implies $a_{i,j} \geq a_{i',j}$, which is impossible because $(i,j) \to (i',j)$ is on L, we have $u(A_{i'}) > u(A_i)$. Then by Lemma 3 and 4 (with (i,j), (i',j), (i',j''), and (i,j'') corresponding to $(1,1)$, $(2,1)$, $(2,2)$, and $(1,2)$ in the case for two candidates, respectively (i.e., $i=1$, $j=1$, $i'=2$, and $j'=2$ in Lemma 3 and Lemma 4), we know that $\Delta(\alpha_2) > 0$ implies that $\Delta(\alpha_1) < 0$, hence a contradiction is derived. □

4 Price of Anarchy for Egoistic Games

A PNE may give a state that is suboptimal in the social welfare for this two-party system. We show a tight bound of 2 on the PoA in the linear link model and the Bradley-Terry model. At the end of this section, we give an upper bound of $1+e$ in the softmax model.

Below is a useful lemma which is used in this section.

Lemma 5. *If i is surpassed by some i', then (i,j) is not a PNE for any $j \in [n]$. Similarly, if j is surpassed by some j', then (i,j) is not a PNE for any $i \in [m]$.*

Proof. If i is surpassed by i', then $i' < i$ and $u(A_{i'}) > u(A_i)$. So $u_A(A_{i'}) > u_A(A_i)$ and $p_{i',j} > p_{i,j}$ for any $j \in [n]$. Thus, $a_{i,j} - a_{i',j} = p_{i,j}u_A(A_i) + (1-p_{i,j})u_A(B_j) - (p_{i',j}u_A(A_{i'}) + (1-p_{i',j})u_A(B_j)) < (p_{i,j} - p_{i',j})(u_A(A_i) - u_A(B_j)) < 0$, which implies that A wants to deviate from i to i'. Then (i,j) is not a PNE. The second part of the claim can be similarly proved. □

We also show a property relating a PNE to an optimal state.

Proposition 1. *Let (i,j) be a PNE and (i^*,j^*) be the optimal state. Then, $u(A_i)+u(B_j) \geq \max\{u(A_{i^*}), u(B_{j^*})\}$.* ■

Proof. Consider the following four cases.

1. If $i \leq i^*$ and $j \leq j^*$, then, $u(A_i)+u(B_j) = u_A(A_i)+u_B(A_i)+u_B(B_j)+u_A(B_j) \geq u_A(A_{i^*})+u_B(A_{i^*}) = u(A_{i^*})$ since $u_A(A_i) \geq u_A(A_{i^*})$ and $u_B(B_j) > u_B(A_{i^*})$. Similarly, we have $u(A_i)+u(B_j) \geq u(B_{j^*})$. Hence, we have $u(A_i)+u(B_j) \geq \max\{u(A_{i^*}), u(B_{j^*})\}$.
2. If $i \leq i^*$ and $u(B_{j^*}) \leq u(B_j)$, then $u(A_i)+u(B_j) \geq u(A_{i^*})$ and $u(A_i)+u(B_j) \geq u(B_j) \geq u(B_{j^*})$ obviously holds. So, we have $u(A_i)+u(B_j) \geq \max\{u(A_{i^*}), u(B_{j^*})\}$.
3. If $u(A_{i^*}) \leq u(A_i)$ and $j \leq j^*$, similarly we obtain $u(A_i)+u(B_j) \geq \max\{u(A_{i^*}), u(B_{j^*})\}$.
4. If $u(A_{i^*}) \leq u(A_i)$ and $u(B_{j^*}) \leq u(B_j)$, obviously we have $u(A_i)+u(B_j) \geq u(A_{i^*})+u(B_{j^*}) \geq \max\{u(A_{i^*}), u(B_{j^*})\}$.

□

4.1 Linear Link Model

In this subsection, we show that, in the linear link model, the PoA of the two-party election game is bounded by 2. We also give an instance to justify that the bound is tight.

Note that for $i \in [m]$, $j \in [n]$,

$$\begin{aligned}
SU_{i,j} &= p_{i,j} \cdot u(A_i) + (1-p_{i,j}) \cdot u(B_j) \\
&= \frac{1+(u(A_i)-u(B_j))/b}{2} \cdot u(A_i) \\
&\quad + \frac{1-(u(A_i)-u(B_j))/b}{2} \cdot u(B_j) \\
&= \frac{1}{2}(u(A_i)+u(B_j)) + \frac{1}{2b}(u(A_i)-u(B_j))^2 \\
&\geq \frac{1}{2}(u(A_i)+u(B_j)).
\end{aligned}$$

and

$$SU_{i,j} = p_{i,j} \cdot u(A_i) + (1 - p_{i,j}) \cdot u(B_j) \leq \max\{u(A_i), u(B_j)\}.$$

Theorem 4. *The two-party election game in the linear link model has the price of anarchy bounded by 2.*

Proof. Let (i, j) be a PNE and (i^*, j^*) be the optimal state. By Lemma 5 we have,

$$\begin{cases} i \text{ is not surpassed by } i^* \\ j \text{ is not surpassed by } j^* \end{cases} \Rightarrow \begin{cases} i \leq i^* \text{ or } u(A_{i^*}) \leq u(A_i) \\ j \leq j^* \text{ or } u(B_{j^*}) \leq u(B_j) \end{cases}$$

Recall that $SU_{i^*,j^*} \leq \max\{u(A_{i^*}), u(B_{j^*})\}$ and $\max\{u(A_{i^*}), u(B_{j^*})\} \leq u(A_i) + u(B_j)$ by Proposition 1. Moreover, $2 \cdot SU_{i,j} \geq u(A_i) + u(B_j)$. Thus, we conclude that $SU_{i,j} \geq SU_{i^*,j^*}/2$, Therefore, the PoA is bounded by 2. ☐

A Tight Example. Let us consider the game instance of $m = n = 2$ in Table 2. Let $\epsilon, \delta > 0$ be two small constants such that $\delta \ll \epsilon \ll b$. In the linear link model, $p_{1,1} = p_{2,2} = 1/2$, $p_{1,2} = (1 - (\epsilon - 2\delta)/b)/2 = 1/2 - (\epsilon - 2\delta)/2b$, $p_{2,1} = (1 + (\epsilon - 2\delta)/b)/2 = 1/2 + (\epsilon - 2\delta)/2b$. Hence,

$$\begin{cases} a_{1,1} = b_{1,1} = \frac{\epsilon}{2}, \\ a_{1,2} = b_{2,1} = \epsilon - \delta \cdot (\frac{1}{2} + \frac{\epsilon - 2\delta}{2b}) \approx \epsilon - \frac{\delta}{2}, \\ a_{2,1} = b_{1,2} = \frac{\epsilon}{2} + \frac{1}{2b} \cdot ((\epsilon - 2\delta)(\epsilon - \delta) - b\delta) \approx \frac{\epsilon}{2} - \frac{\delta}{2}, \end{cases}$$

the last terms of the last two equations above follow from $\epsilon, \delta \ll b$. It is easy to see that $(1, 1)$ is the only PNE and $(2, 2)$ is the optimal state. Thus, the PoA of the instance is approximately

$$2 - \frac{2\delta}{\epsilon},$$

which is close to 2 as δ/ϵ approaches 0.

A		B	
$u_A(A_i)$	$u_B(A_i)$	$u_B(B_j)$	$u_A(B_j)$
ϵ	0	ϵ	0
$\epsilon - \delta$	$\epsilon - \delta$	$\epsilon - \delta$	$\epsilon - \delta$

	B_1	B_2
A_1	$a_{1,1},\ b_{1,1}$	$a_{1,2},\ b_{1,2}$
A_2	$a_{2,1},\ b_{2,1}$	$a_{2,2},\ b_{2,2}$

$\approx$

	B_1	B_2
A_1	$\frac{\epsilon}{2},\ \frac{\epsilon}{2}$	$\epsilon - \frac{\delta}{2},\ \frac{\epsilon}{2} - \frac{\delta}{2}$
A_2	$\frac{\epsilon}{2} - \frac{\delta}{2},\ \epsilon - \frac{\delta}{2}$	$\epsilon - \delta,\ \epsilon - \delta$

Table 2. An example illustrating a lower bound on the worst PoA of the egoistic game in both the linear link model and the softmax model ($m = n = 2$, $0 < \epsilon, \delta \ll b$, and $\delta \ll \epsilon$).

4.2 Bradley-Terry Model

Though in Section 3.1, we show that a PNE does not always exist in the Bradley-Terry model, one may be curious about how good or bad its PoA is once it has a PNE. In this subsection, we show that its price of anarchy is bounded by 2, which is not tight, in this model once a PNE exists in the game. A lower bound on the PoA is also given.

Theorem 5. *The two-party election game in the Bradley-Terry model has the price of anarchy bounded by 2.*

Proof. Let (i, j) be a PNE and (i^*, j^*) be the optimal state. Note that $SU_{i^*,j^*} = p_{i^*,j^*} \cdot u(A_{i^*}) + (1 - p_{i^*,j^*}) \cdot u(B_{j^*}) \leq \max\{u(A_{i^*}), u(B_{j^*})\}$. Moreover, the Cauchy-Schwarz inequality implies that $u(A_i)^2 + u(B_j)^2 \geq (u(A_i) + u(B_j))^2/2$. Hence, we obtain that

$$\begin{aligned} SU_{i,j} &= \frac{u(A_i)}{u(A_i) + u(B_j)} \cdot u(A_i) + \frac{u(B_j)}{u(A_i) + u(A_j)} \cdot u(B_j) \\ &= \frac{u(A_i)^2 + u(B_j)^2}{u(A_i) + u(B_j)} \geq \frac{1}{2} \cdot (u(A_i) + u(B_j)). \end{aligned}$$

Together, by Proposition 1 that $u(A_i) + u(B_j) \geq \max\{u(A_{i^*}), u(B_{j^*})\}$, we finally have that $SU_{i,j} \geq SU_{i^*,j^*}/2$, Thus, the PoA is bounded by 2. □

A Lower Bound Example. Consider the instance in Table 3. Let $\epsilon, \delta > 0$ be two real numbers such that $\delta \ll \epsilon$. We have $p_{1,1} = p_{2,2} = 1/2$, $p_{1,2} = \epsilon/(3\epsilon - 2\delta)$, $p_{2,1} = 2(\epsilon - \delta)/(3\epsilon - 2\delta)$. First, $a_{1,1} = b_{1,1} = \epsilon/2$ and $a_{2,2} = b_{2,2} = \epsilon - \delta$. Second, $a_{2,1} = b_{1,2} = 2(\epsilon - \delta)^2/(3\epsilon - 2\delta) < 2(\epsilon - \delta)^2/(3(\epsilon - \delta)) = 2(\epsilon - \delta)/3$. Since $\delta < \epsilon/8$ by $\delta \ll \epsilon$, we have $a_{2,1} = b_{1,2} > (2\epsilon^2 - 4\delta\epsilon)/3\epsilon > \epsilon/2$. Third, $a_{1,2} = b_{2,1} = (\epsilon^2 + 2(\epsilon - \delta)^2)/(3\epsilon - 2\delta)$, which is less than $(\epsilon^2 - \delta\epsilon + \delta^2)/(\epsilon - \delta) = \epsilon + \delta^2/(\epsilon - \delta)$, and is equal to $(3(\epsilon - \delta)^2 + 2\delta\epsilon - \delta^2)/(3(\epsilon - \delta) + \delta) = ((\epsilon - \delta) \cdot (3(\epsilon - \delta) + \delta) + \epsilon\delta - \delta^2)/(3(\epsilon - \delta) + \delta) = (\epsilon - \delta) + (\delta\epsilon - \delta^2)/(3\epsilon - 2\delta)$, which is greater than $(\epsilon - \delta)$. Then states $(1, 2)$ and $(2, 1)$ are both PNE. By letting δ be arbitrarily close to 0, we derive that the PoA is bounded by

$$\frac{2(\epsilon - \delta)}{(\epsilon^2 + 2(\epsilon - \delta)^2)/(3\epsilon - 2\delta) + 2(\epsilon - \delta)^2/(3\epsilon - 2\delta)} \approx \frac{2\epsilon}{5\epsilon^2/3\epsilon} = \frac{6}{5}.$$

A		B	
$u_A(A_i)$	$u_B(A_i)$	$u_B(B_j)$	$u_A(B_j)$
ϵ	0	ϵ	0
$\epsilon - \delta$	$\epsilon - \delta$	$\epsilon - \delta$	$\epsilon - \delta$

	B_1	B_2
A_1	$a_{1,1},\ b_{1,1}$	$a_{1,2},\ b_{1,2}$
A_2	$a_{2,1},\ b_{2,1}$	$a_{2,2},\ b_{2,2}$

=

	B_1		B_2	
A_1	$\frac{\epsilon}{2}$,	$\frac{\epsilon}{2}$	$(\epsilon - \delta, \epsilon + \frac{\delta^2}{\epsilon - \delta})$,	$(\frac{\epsilon}{2}, \frac{2(\epsilon - \delta)}{3})$
A_2	$(\frac{\epsilon}{2}, \frac{2(\epsilon - \delta)}{3})$,	$(\epsilon - \delta, \epsilon + \frac{\delta^2}{\epsilon - \delta})$	$\epsilon - \delta$,	$\epsilon - \delta$

Table 3. An example illustrating a lower bound 6/5 on the worst PoA of the egoistic game in the Bradley-Terry model ($m = n = 2$, $0 < \delta \ll \epsilon$).

4.3 Softmax Model

In this subsection, we show that the price of anarchy of the game is bounded by $1 + e$, which is not tight, in the softmax model. A lower bound on the PoA is also given.

Note that $p_{i,j} \geq e^0/(e^0 + e) \geq 1/(1 + e)$ and $p_{i,j} \leq e/(e^0 + e) \leq e/(1 + e)$ for any i, j.

$$\begin{aligned} SU_{i,j} &= p_{i,j} \cdot u(A_i) + (1 - p_{i,j}) \cdot u(B_j) \\ &\geq \min\{p_{i,j}, 1 - p_{i,j}\} \cdot (u(A_i) + u(B_j)) \\ &\geq \frac{1}{1 + e}(u(A_i) + u(B_j)), \end{aligned}$$

and

$$\begin{aligned}SU_{i,j} &= p_{i,j} \cdot u(A_i) + (1 - p_{i,j}) \cdot u(B_j) \\ &\leq \frac{e}{1+e} \cdot \max\{u(A_i), u(B_j)\} + \frac{1}{1+e} \cdot \min\{u(A_i), u(B_j)\}.\end{aligned}$$

Now, we are ready for Theorem 6 and its proof.

Theorem 6. *The two-party election game in the softmax model has the PoA bounded by* $1 + e$.

Proof. Let (i, j) be a PNE and (i^*, j^*) be the optimal state. By Lemma 5 we have,

$$\begin{cases} i \text{ is not surpassed by } i^* \\ j \text{ is not surpassed by } j^* \end{cases} \Rightarrow \begin{cases} i \leq i^* \text{ or } u(A_{i^*}) \leq u(A_i) \\ j \leq j^* \text{ or } u(B_{j^*}) \leq u(B_j) \end{cases}$$

Recall that $(1 + e) \cdot SU_{i,j} \geq (u(A_i) + u(B_j))$. Without loss of generality, let $\max\{u(A_{i^*}), u(B_{j^*})\} = u(A_{i^*})$. Consider the following four cases.

1. $i \leq i^*$ and $j \leq j^*$. Then,

$$u(A_i) + u(B_j) - \left(\frac{e}{1+e} \cdot u(A_{i^*}) + \frac{1}{1+e} \cdot u(B_{j^*})\right) \geq 0$$

   since $u_A(A_i) \geq (e/(1+e))u_A(A_{i^*}) + (1/(1+e))u_A(B_{j^*})$ and $u_B(B_j) \geq (e/(1+e))u_B(A_{i^*}) + (1/(1+e))u_B(B_{j^*})$.
2. $i \leq i^*$ and $u(B_{j^*}) \leq u(B_j)$. Then

$$\begin{aligned}&u(A_i) + u(B_j) - \left(\frac{e}{1+e} \cdot u(A_{i^*}) + \frac{1}{1+e} \cdot u(B_{j^*})\right) \\ &\geq (u_A(A_i) + u_B(A_i)) + \frac{e}{1+e}(u_B(B_j) + u_A(B_j)) \\ &\quad - \frac{e}{1+e}(u_A(A_{i^*}) + u_B(A_{i^*})) \\ &\geq 0,\end{aligned}$$

   in which the last inequality holds since $u_A(A_i) \geq u_A(A_{i^*})$ and $u_B(B_j) > u_B(A_{i^*})$.
3. $u(A_{i^*}) \leq u(A_i)$ and $j \leq j^*$. This case is similar to (2).
4. $u(A_{i^*}) \leq u(A_i)$ and $u(B_{j^*}) \leq u(B_j)$. Obviously,

$$\begin{aligned}&u(A_i) + u(B_j) - \left(\frac{e}{1+e} \cdot u(A_{i^*}) + \frac{1}{1+e} \cdot u(B_{j^*})\right) \\ &\geq \frac{1}{1+e} \cdot u(A_i) + \frac{e}{1+e} \cdot u(B_j) \geq 0.\end{aligned}$$

We conclude that $SU_{i,j} \geq SU_{i^*,j^*}/(1 + e)$. Therefore, the PoA is bounded by $1 + e$. □

A Lower Bound Example. Consider the instance in Table 2. We have $p_{1,1} = p_{2,2} = 1/2$, $p_{1,2} = e^{\epsilon}/(e^{\epsilon} + e^{2\epsilon-2\delta}) \approx 1/2$, $p_{2,1} = e^{2\epsilon-2\delta}/(e^{\epsilon} + e^{2\epsilon-2\delta}) \approx 1/2$. Hence, $a_{1,1} = b_{1,1} = \epsilon/2$, $a_{2,1} = b_{1,2} \approx (\epsilon - \delta)/2$, and $a_{2,2} = b_{2,2} = \epsilon - \delta$. Similar to the analysis in Sect. 4.1, we obtain that the price of anarchy of this instance is approximately $2 - \frac{2\delta}{\epsilon}$, which is close to 2 as δ/ϵ approaches 0.

5 Non-Egoistic Games

Without the egoistic property, a game instance with no PNE in the linear link model can be constructed, and another game instance with an unbounded PoA can also be given in the three models.

5.1 No Existence Guarantee of PNE in the Linear Link Model

Consider the two-party election game in the linear link model. As the instance illustrated in Table 4, it is *not* an egoistic one (e.g., $u_B(A_1) > u_B(B_2)$). We derive that $p_{1,1} = (1 + (60 - 100)/100)/2 = 0.3$, $p_{1,2} = (1+(60-25)/100)/2 = 0.675$, $p_{2,1} = (1+(25-100)/100)/2 = 0.125$, and $p_{2,2} = (1+(25-25)/100)/2 = 0.5$. Hence, we obtain the payoff matrix as illustrated in the bottom of Table 1.

A		B	
$u_A(A_i)$	$u_B(A_i)$	$u_B(B_j)$	$u_A(B_j)$
50	10	10	90
5	20	5	20

	B_1	B_2
A_1	$a_{1,1}$, $b_{1,1}$	$a_{1,2}$, $b_{1,2}$
A_2	$a_{2,1}$, $b_{2,1}$	$a_{2,2}$, $b_{2,2}$

=

	B_1	B_2
A_1	78, 10	40.25, 8.375
A_2	79.375, 11.25	12.5, 12.5

Table 4. An example having no PNE in the linear link model ($m = n = 2$, $b = 100$).

From the bottom of Table 4, none of the states is a PNE (e.g., in state $(1, 1)$, A wants to deviate from his strategy to 2 because he or she will get the payoff 79.375 which is better than 78).

5.2 Unbounded PoA in the Three Models

In this subsection, we show that the two-party election game has unbounded PoA if it is *not* egoistic. Let us consider the game instance illustrated in Table 5. Its payoff matrix is different and we will show that its PoA is unbounded in the linear link, softmax, and Bradley-Terry models.

A		B	
$u_A(A_i)$	$u_B(A_i)$	$u_B(B_j)$	$u_A(B_j)$
ϵ	0	ϵ	0
0	b	0	b

Table 5. An illustrating non-egoistic game instance for $m = n = 2$.

The Linear Link Model Let $\epsilon > 0$ be a small constant. We derive that $p_{1,1} = p_{2,2} = 1/2$, $p_{1,2} = (1 + (\epsilon - b)/b)/2$, and $p_{2,1} = (1 + (b - \epsilon)/b)/2$. Hence, we obtain the payoff matrix as illustrated in Table 6. Clearly, state $(1, 1)$ is a PNE, so the PoA is at least b/ϵ, which is unbounded as ϵ approaches to 0.

The Softmax Model By definition of the softmax model, we derive that $p_{1,1} = p_{2,2} = 1/2$, $p_{1,2} = e^{\epsilon/b}/(e^{\epsilon/b} + e)$, and $p_{2,1} = e/(e^{\epsilon/b} + e)$. Hence, we obtain the payoff matrix as illustrated in the bottom of Table 7. Clearly, state $(1, 1)$ is a PNE, so the PoA is at least

$$\frac{b}{2\epsilon e^{\epsilon}/(e^{\epsilon}+1)},$$

which is unbounded as ϵ approaches to 0.

	B_1	B_2
A_1	$a_{1,1},\ b_{1,1}$	$a_{1,2},\ b_{1,2}$
A_2	$a_{2,1},\ b_{2,1}$	$a_{2,2},\ b_{2,2}$

=

	B_1	B_2
A_1	$\frac{\epsilon}{2},\ \frac{\epsilon}{2}$	$b - \frac{\epsilon(b-\epsilon)}{2b},\ 0$
A_2	$0,\ b - \frac{\epsilon(b-\epsilon)}{2b}$	$\frac{b}{2},\ \frac{b}{2}$

Table 6. The payoff matrix of instance in Table 5 in the linear link model.

	B_1	B_2
A_1	$a_{1,1},\ b_{1,1}$	$a_{1,2},\ b_{1,2}$
A_2	$a_{2,1},\ b_{2,1}$	$a_{2,2},\ b_{2,2}$

=

	B_1	B_2
A_1	$\frac{\epsilon e^{\epsilon}}{e^{\epsilon}+1},\ \frac{\epsilon e^{\epsilon}}{e^{\epsilon}+1}$	$\frac{\epsilon e^{\epsilon}+eb}{e^{\epsilon}+1},\ 0$
A_2	$0,\ \frac{\epsilon e^{\epsilon}+eb}{e^{\epsilon}+1}$	$\frac{b}{2},\ \frac{b}{2}$

Table 7. The payoff matrix of instance in Table 5 in the softmax model.

The Bradley-Terry Model By definition of the Bradley-Terry model, we derive that $p_{1,1} = p_{2,2} = 1/2$, $p_{1,2} = \epsilon/(\epsilon + b)$, and $p_{2,1} = b/(\epsilon + b)$. Hence, we obtain the payoff matrix as illustrated in the bottom of Table 8. Clearly, state $(1, 1)$ is a PNE, so the PoA is at least b/ϵ, which is unbounded as ϵ approaches to 0.

	B_1	B_2
A_1	$a_{1,1},\ b_{1,1}$	$a_{1,2},\ b_{1,2}$
A_2	$a_{2,1},\ b_{2,1}$	$a_{2,2},\ b_{2,2}$

=

	B_1	B_2
A_1	$\frac{\epsilon}{2},\ \frac{\epsilon}{2}$	$\frac{\epsilon^2+b^2}{b+\epsilon},\ 0$
A_2	$0,\ \frac{\epsilon^2+b^2}{b+\epsilon}$	$\frac{b}{2},\ \frac{b}{2}$

Table 8. The payoff matrix of instance in Table 5 in the Bradley-Terry model.

6 Conclusions and Future Work

We summarize our results in Table 9. From the perspective of price of anarchy, the two-party election game is "good" in the sense that its price of anarchy is constantly bounded if the game is egoistic. Based on the three investigated models, the game may not have a PNE if it is not egoistic, though according to our simulation, we conjecture that the game in the softmax model always has a PNE.

In this paper, we only focus on pure Nash equilibria. The other equilibrium concepts, such as mixed Nash equilibria, approximate Nash equilibria, etc., also deserve further investigations. We conjecture that the two-party election game is not a smooth game, and different equilibrium concepts may have different tight bounds on the corresponding price of anarchy.

There is a still a gap between the upper bound and lower bound on the worst PoA of the egoistic two-party election game in the softmax model and the Bradley-Terry model as well. By examining the game instances randomly sampled, we conjecture that the upper bound is at most 2 in the softmax model and strictly below 2 in the Bradley-Terry model.

It will be interesting to generalize our results to election games of two or more parties, and to see if our results are robust in more general settings for modeling the winning probabilities. Coalition between the parties and thus strong equilibria can also be considered. Moreover, to design voting mechanisms toward settings for the linear link model or the softmax model is another interesting direction.

	Linear Link	Bradley-Terry	Softmax
PNE w/ egoism	✓	×	✓
PNE w/o egoism	×	×	?#
PoA upper bound w/ egoism	2*	2	$1+e$
PoA lower bound w/ egoism	2	6/5	2
Worst PoA w/o egoism	∞	∞	∞

Table 9. *: the bound is tight. ∞: unbounded. ✓: PNE always exists. ×: PNE does NOT always exist. #: Based on our simulation results, we conjecture that the game in the softmax model always has a PNE (even without egoism).

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