Optimal Designs for Accelerated Degradation Tests

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Outline

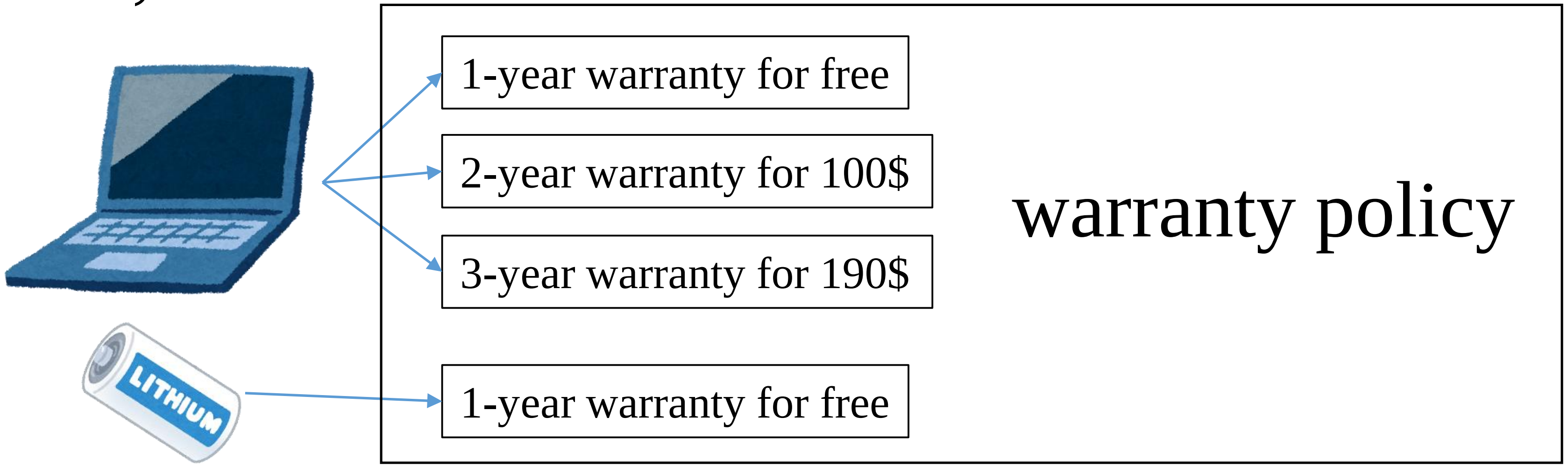
- Introduction
- Accelerated degradation tests and optimal designs
- Optimal design for exponential dispersion accelerated degradation tests
  - A single accelerating variable
  - Two accelerating variables
    - Without interaction
    - With interaction
- Conclusions

What is reliability?

- Reliability is quality over time.
- A formal definition of reliability is given as:
  Reliability is the probability that a product will operate or a service will be provided properly for a specified period of time (design life) under the design operating conditions (such as temperature, load, volt…) without failure.
- Let T be a random variable that denotes the product operating time before failure. Then, the reliability of a product is defined as
  $$R(t) = 1 - F_T(t) = P(T \geq t)$$
  where $F_T(t)$ is the cumulative distribution function of T.

Why reliability is important?

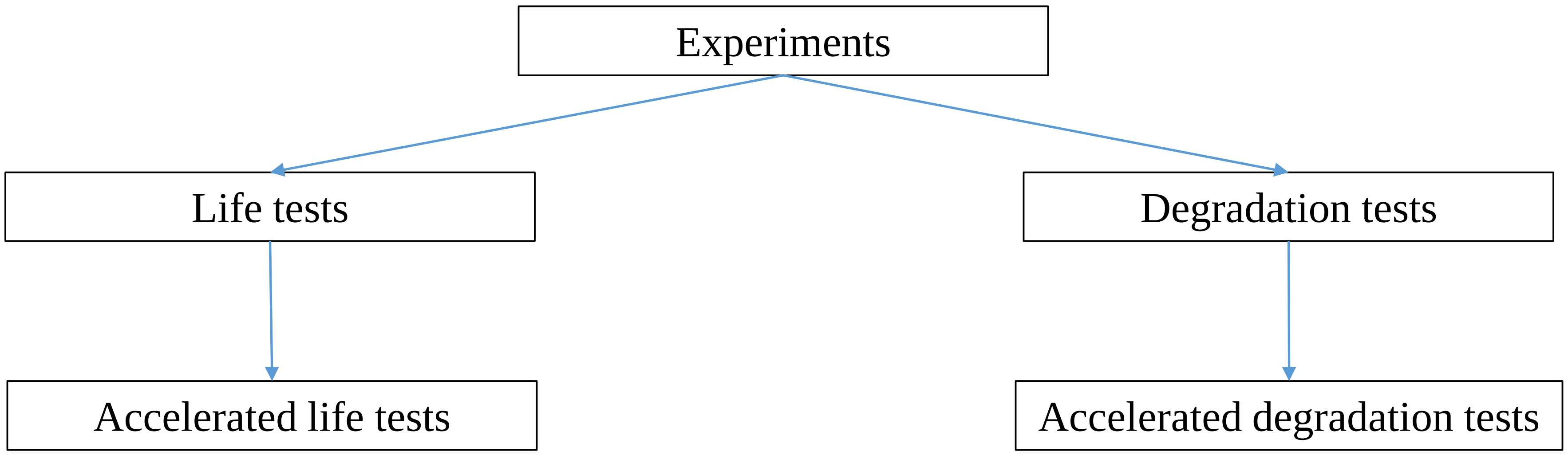
- Customers expect purchased products to be reliable and safe.
- Predicting product warranty costs.
- Comparing components from two or more different manufacturers, materials, and so on.



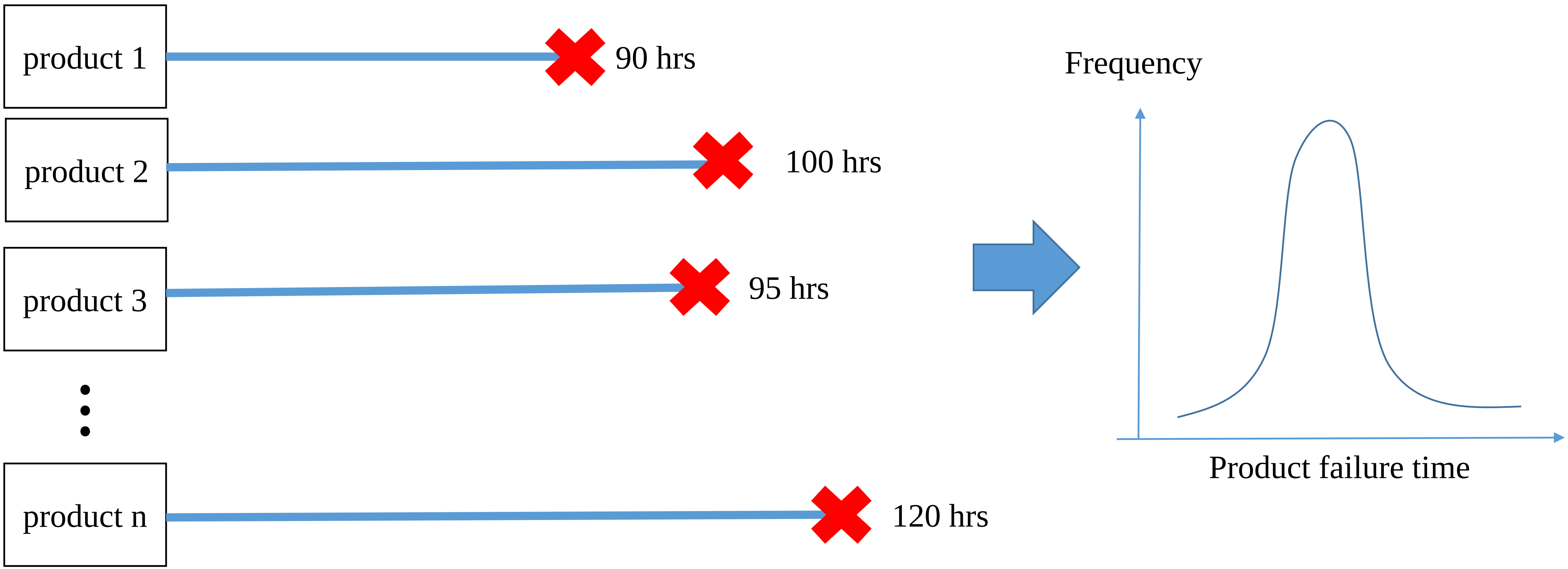
How to assess reliability

- In most cases, this will involve the collection of reliability data from studies such as laboratory tests (or designed experiments) of products, tests on early prototype units, careful monitoring of early-production units in the field, analysis of warranty data, and systematic longer-term tracking of products in the field.

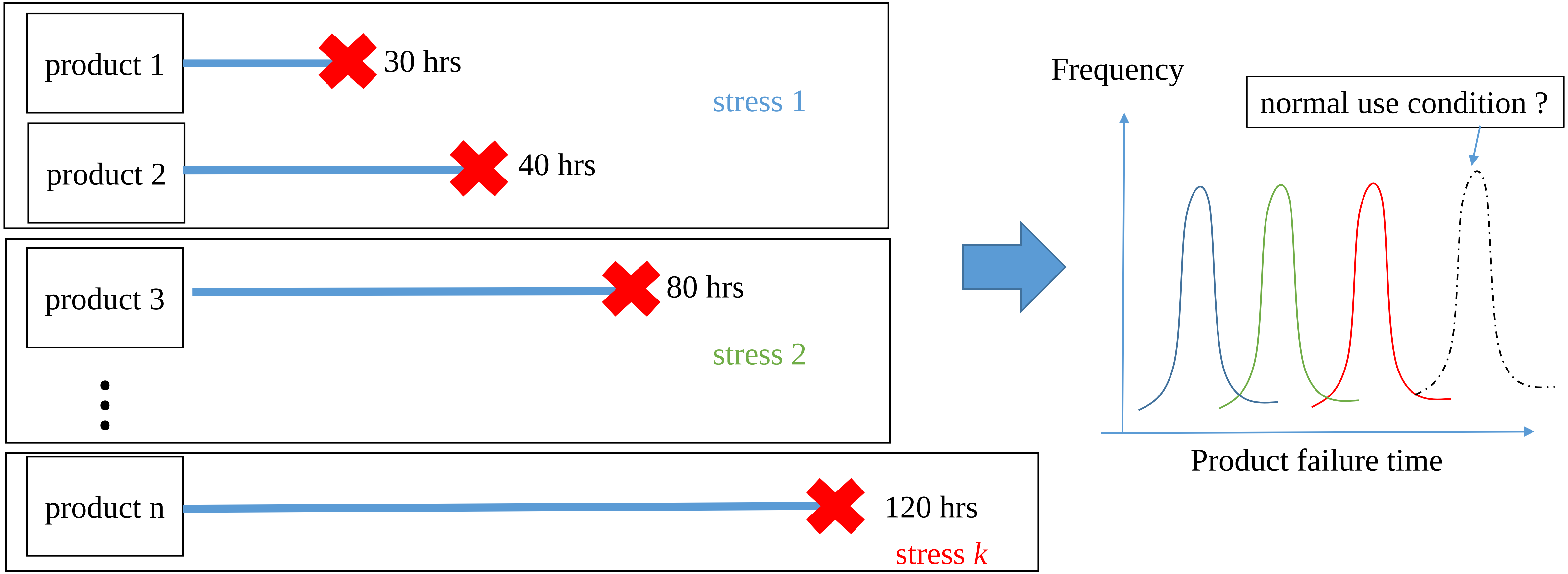
Collection of reliability data



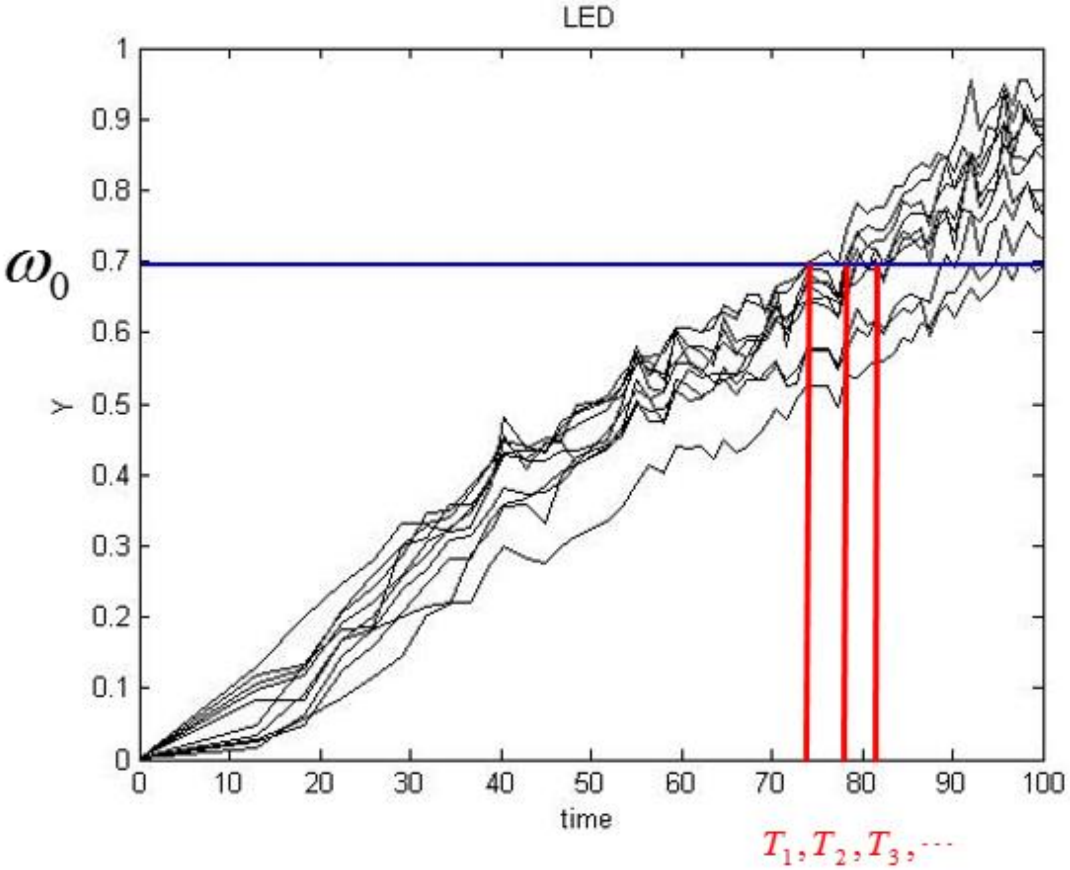
Life tests


product 1
90 hrs
product 2
100 hrs
product 3
95 hrs
product n
120 hrs
Frequency
Product failure time

Accelerated life tests

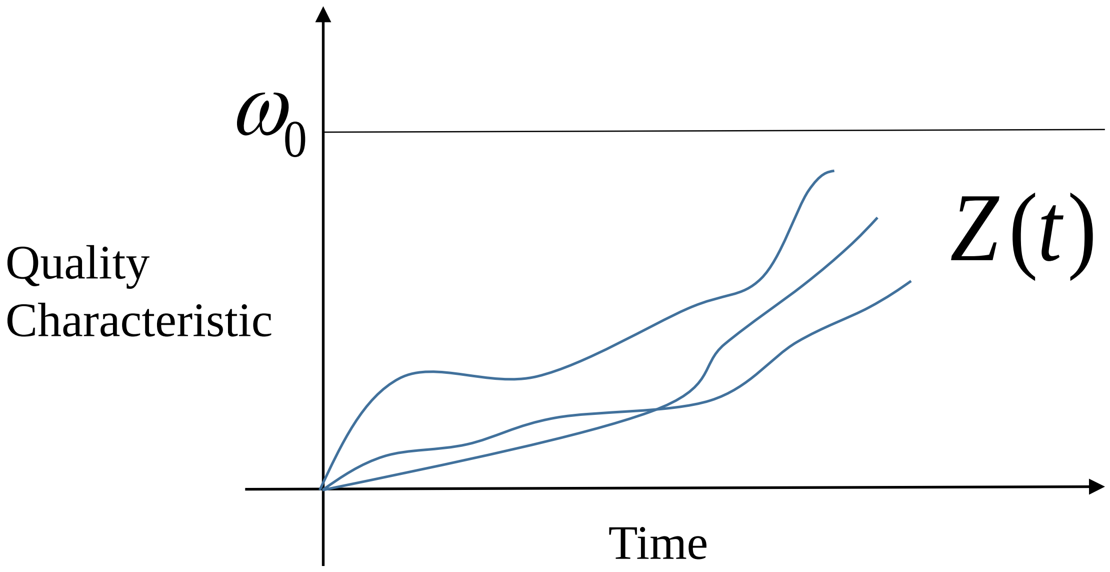


Degradation tests



$$T = \inf \left\{ t \geq 0 \mid Z\left(t\right) \geq \omega_0 \right\}$$

Degradation tests



time	product 1	product 2	…	product n
t_1	$Z_1(t_1)$	$Z_2(t_1)$		$Z_n(t_1)$
t_2	$Z_1(t_2)$	$Z_2(t_2)$		$Z_n(t_2)$
t_3	$Z_1(t_3)$	$Z_2(t_3)$		$Z_n(t_3)$
⋮				
t_m	$Z_1(t_m)$	$Z_2(t_m)$		$Z_n(t_m)$

$$T = \inf\left\{t \geq 0 \mid Z(t) \geq \omega_0\right\}$$

Degradation tests

- General path models

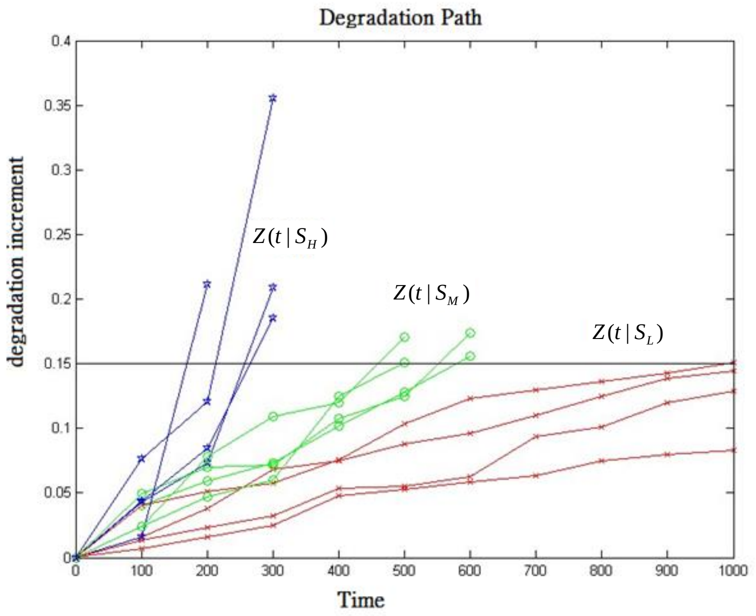
$$Z(t) = G(t) + \epsilon_t$$

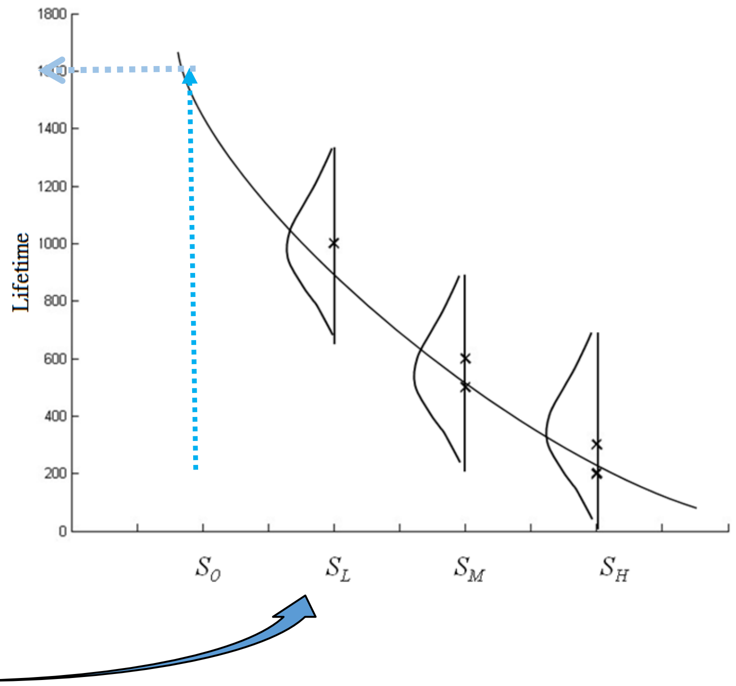
where $G(t)$ is a deterministic trend and ϵ_t is independently distributed.

- Stochastic processes (Lévy process)
  $Z(t)$ has the following properties:
  - $Z(t)$ has continuous path and $Z(0) = 0$.
  - $Z(t)$ has independent increment.
    That is, $Z(t_i) - Z(t_{i-1})$ is independent with $Z(t_j) - Z(t_{j-1})$ for $t_j < t_{i-1}$.
  - $Z(t) - Z(s)$ is equal in distribution to Z_{t-s} for any $s < t$.

Accelerated degradation tests (ADT)

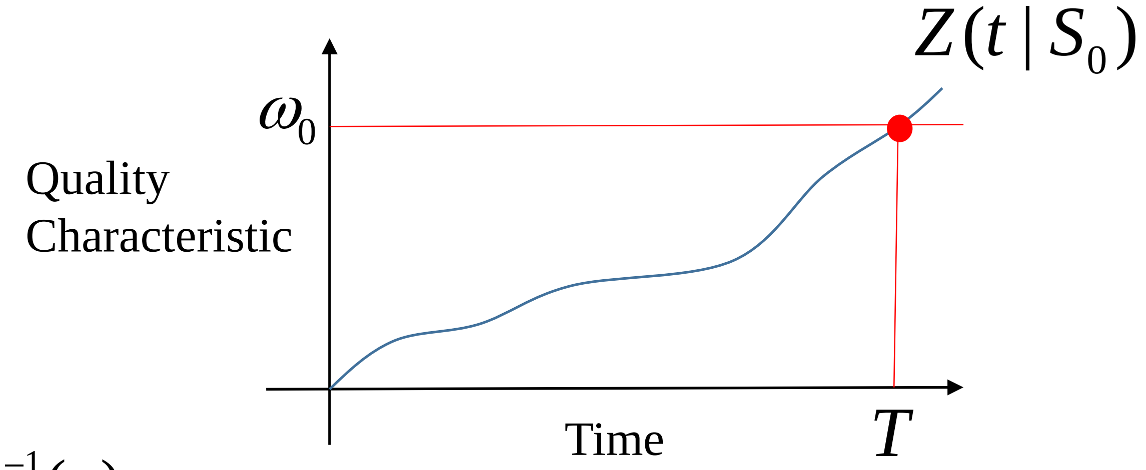
- Use the degradation data under higher stress levels to extrapolate the product's lifetime distribution under normal use condition.





How to predict product lifetime based on ADTs data?

- $Z(t|S_0)$: Degradation path under normal use condition
- Normal use condition: S_0
- Threshold: ω_0
- $T = \inf\{t \mid Z(t \mid S_0) \geq \omega_0\}$
- $F_T(t) = P(T \leq t)$
- The q quantile of product lifetime: $\xi_q = F_T^{-1}(q)$
  Mean time to failure (MTTF) : $E(T)$



Exponential dispersion degradation model

- $Z(t) \sim ED(\mu t, \lambda)$

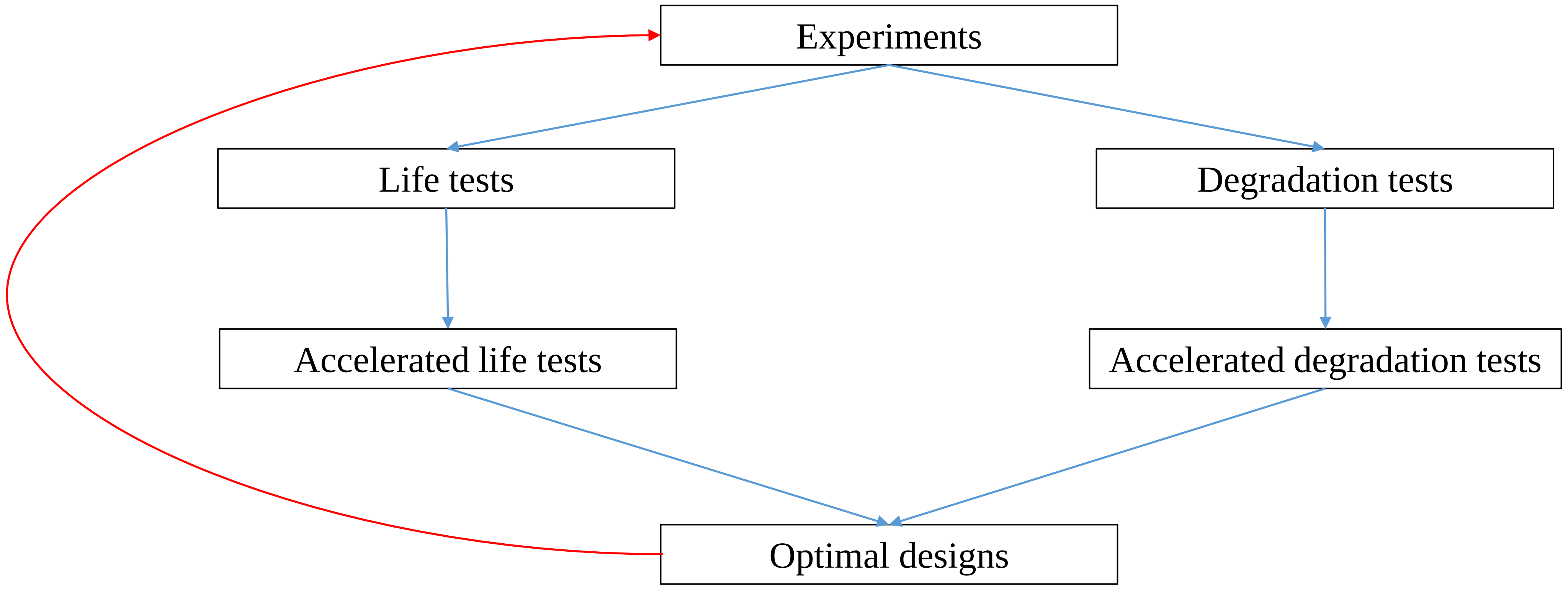
  $\Delta Z_j = Z(t_j) - Z(t_{j-1})$ has the probability density function:

$$f(\Delta z_j \mid \mu, \lambda) = c(\Delta z_j \mid \lambda, \Delta t_j) e^{\lambda\{\Delta z_j \varpi(\mu) - \Delta t_j \kappa[\varpi(\mu)]\}}$$

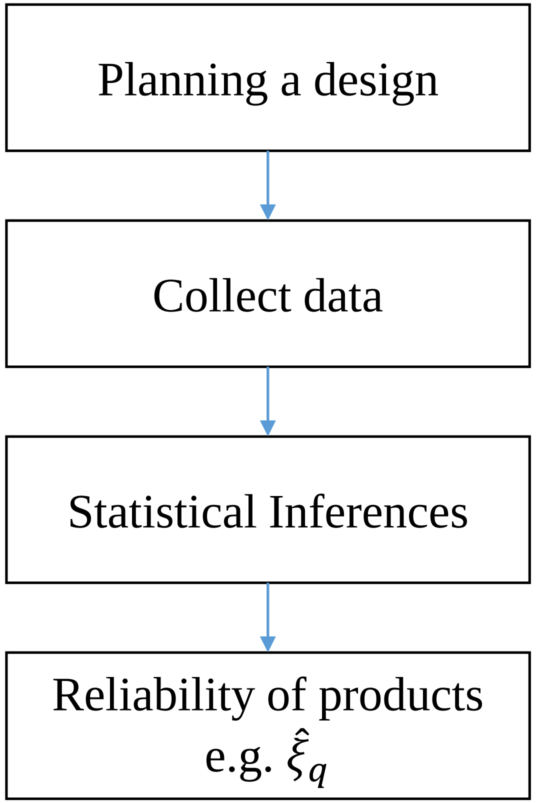
- $E(Z(t)) = \mu t,\ \mathrm{Var}(Z(t)) = V(\mu) t / \lambda$
- $V(\mu) = \mu^d,\ d \in (-\infty, 0] \cup [1, \infty)$

d	*d*=0	*d*=1	1<*d*<2	*d*=2	*d*=3
distribution	Wiener	Poisson	Compound Poisson	Gamma	Inverse Gaussian

Collection of reliability data

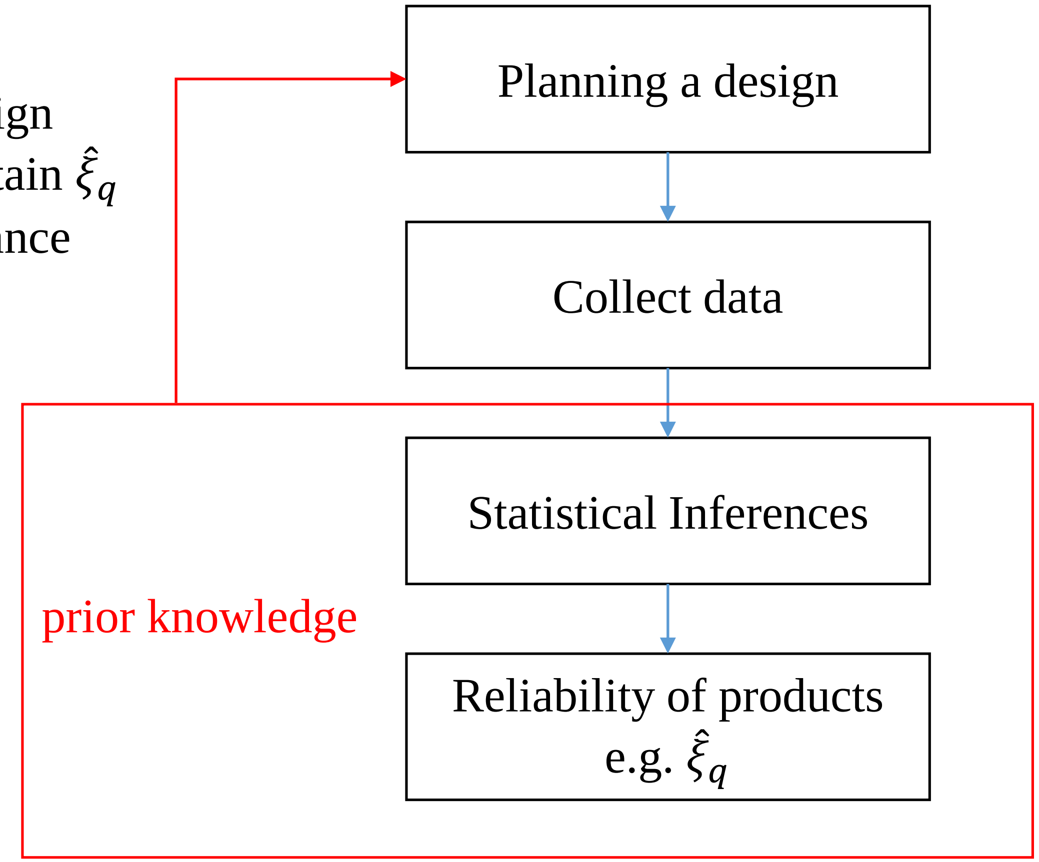


Analysis procedure for laboratory data



Analysis procedure for laboratory data

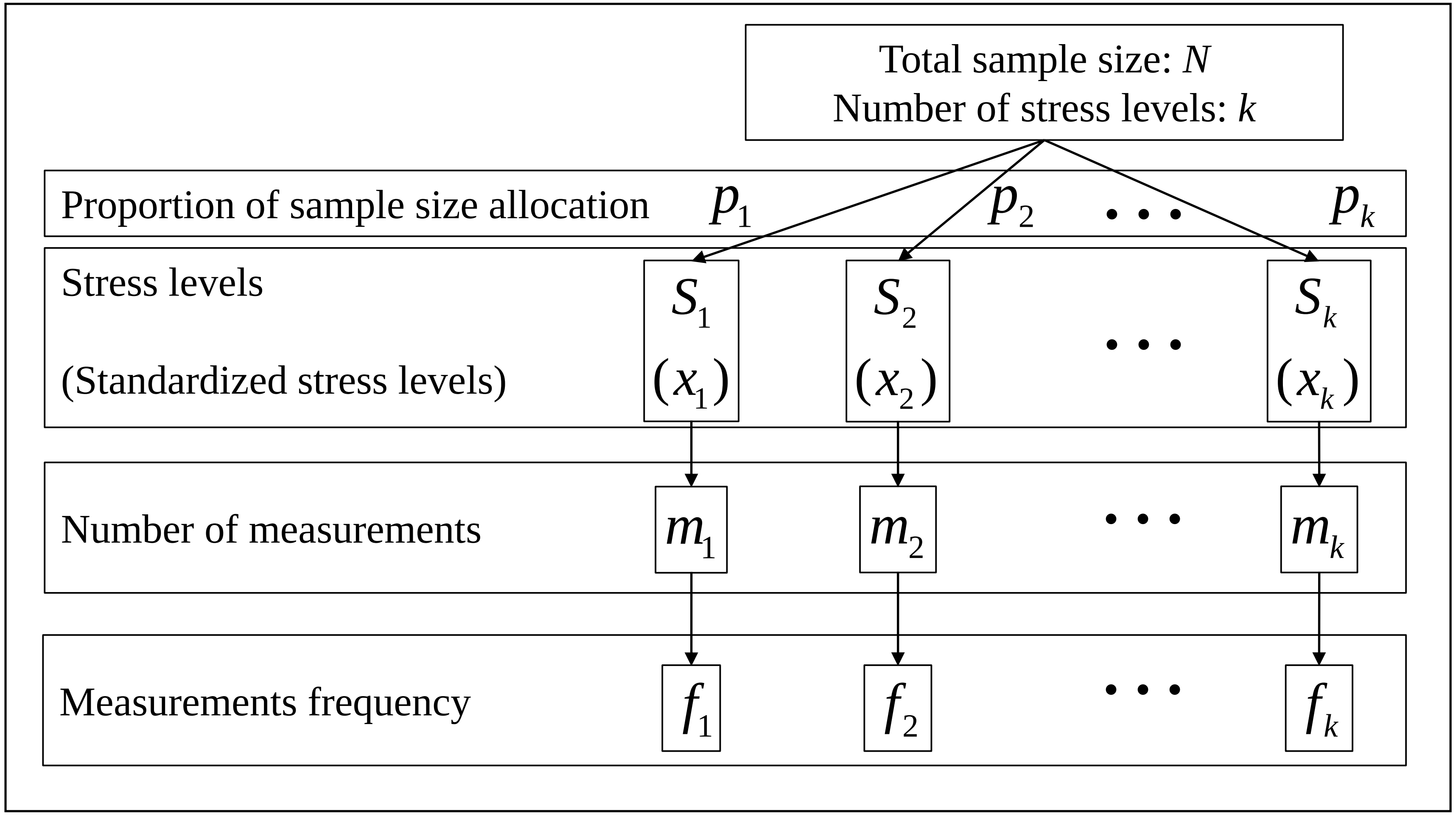
Find an optimal design such that we can obtain $\hat{\xi}_q$ with minimum variance



Optimal criteria

- D-optimality: minimize the determinant of the covariance matrix of parameters.
- A-optimality: minimize the trace of the covariance matrix of parameters.
- E-optimality: minimize the maximum eigenvalue of the covariance matrix of parameters.
- V-optimality: minimize the variance of $\hat{\xi}_q$.

Layout of a k-level ADT



$$\zeta = \begin{pmatrix} x_1 & \cdots & x_k \\ n_1 & \cdots & n_k \\ m_1 & \cdots & m_k \\ f_1 & \cdots & f_k \end{pmatrix}$$

$$n_l = N \times p_l$$

Goal of designing an ADT plan

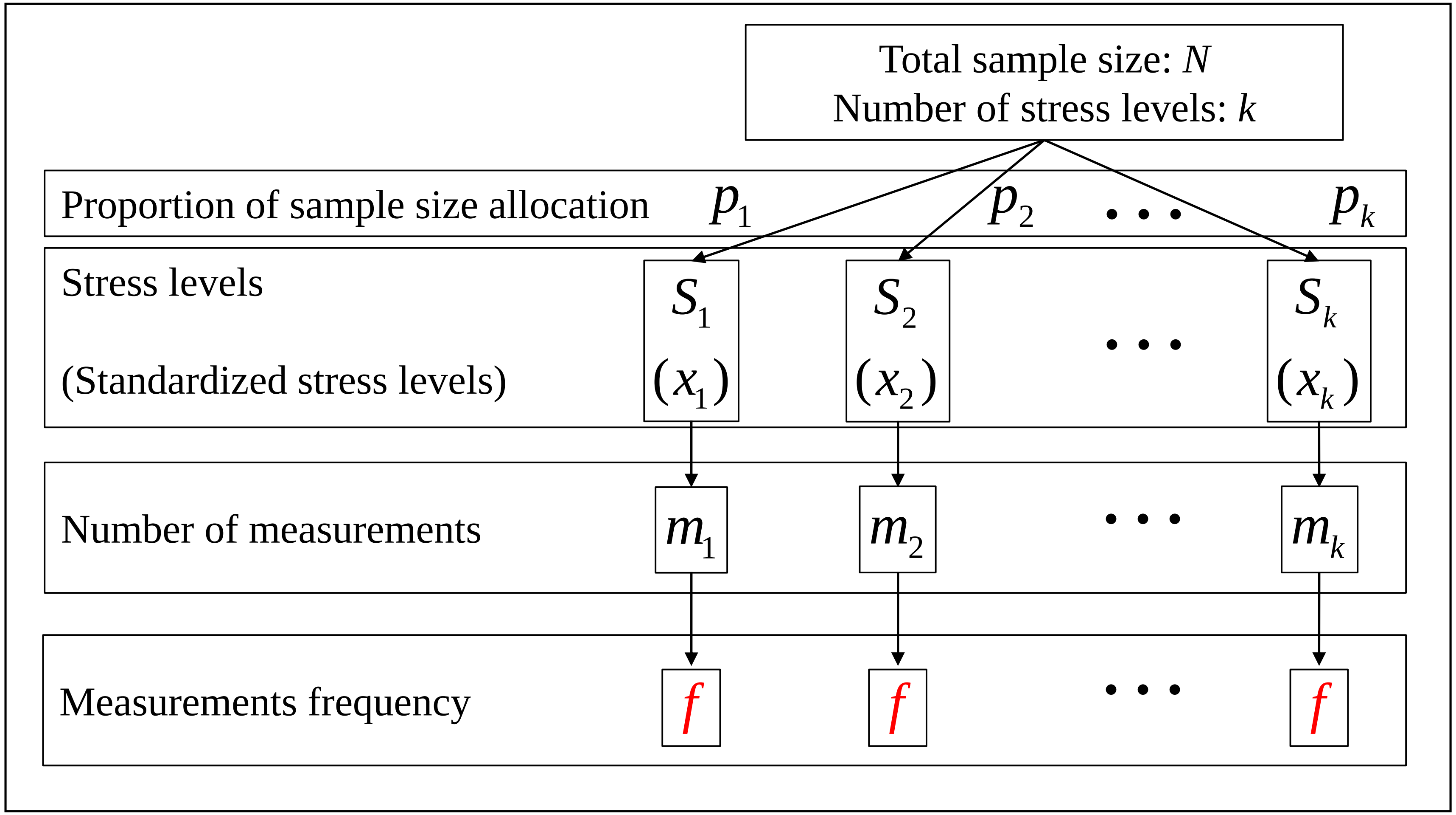
- The goal of this study is to find an optimal design which minimizes the asymptotic variance of ξ_q (MLE). That is,

$$\zeta^* = \underset{\zeta}{\operatorname{argmin}}\, \mathrm{AVar}(\xi_q \mid \zeta).$$

- I will use the exponential dispersion accelerated degradation model to illustrate the procedure.

- Optimal design for EDADTs of a single accelerating variable
    - Problem formulation
    - The expression of Avar$(\hat{\xi}_q)$
    - Conjecture V-optimal design
    - Optimal allocation rule based on cost constraint

Problem formulation



$$\zeta = \begin{pmatrix} x_1 & \cdots & x_k \\ n_1 & \cdots & n_k \\ m_1 & \cdots & m_k \end{pmatrix}$$

Problem formulation

- Let $Z_i(t_{jl}|x_l)$ $(i=1,\ldots,n_l,\ j=1,\ldots m_l,\ l=1,\ldots,k)$ denote the degradation of ith test unit at time $t_{jl}=j\times f\times\Delta t$ under lth stress-level x_l.

- $Z_i(t_{jl}|x_l)\sim ED(\mu(x_l)t_{jl},\lambda),\ \ln(\mu(x_l))=a+bx_l, b>0,\ x_l\in[x_L,1]$
  $\Delta Z_{ijl}=Z_i(t_{jl}|x_l)-Z_i(t_{(j-1)l}|x_l)$ has the probability density function:

$$f(\Delta z_{ijl}|\mu(x_l),\lambda)=c(\Delta z_{ijl}|\lambda,\Delta t_{jl})e^{\lambda\{\varpi(\mu(x_l))\Delta z_{ijl}-\Delta t_{jl}\kappa[\varpi(\mu(x_l))]\}}$$

Maximum likelihood estimation

- The likelihood function

$$L(a,b,\lambda) = \prod_{l=1}^{k} \prod_{i=1}^{n_l} \prod_{j=1}^{m_l} f\left(\Delta z_{ijl} | \mu(x_l), \lambda\right).$$

- The log-likelihood function

$$l(a,b,\lambda) = C + \sum_{l=1}^{k} \sum_{i=1}^{n_i} \sum_{j=1}^{m_l} \lambda \varpi\big(\mu(x_l)\big) \Delta \mathrm{z}_{ijl} - \lambda \sum_{l=1}^{k} \kappa\Big(\varpi\big(\mu(x_l)\big)\Big) n_l m_l \Delta t\,.$$

- Maximum likelihood estimation

$$\big(\hat{a}, \hat{b}, \hat{\lambda}\big) = \mathrm{argmax}\, l(a,b,\lambda).$$

Properties of MLE

- Fisher information matrix

$$I^*(\theta|\zeta) = E\left[-\frac{\partial^2 l(\theta)}{\partial\theta_i \partial\theta_j}\right] = \begin{pmatrix} \lambda\Delta t e^{-a(d-2)} I(\theta|\zeta) & 0^T \\ 0 & \frac{1}{2\lambda^2}\sum_{l=1}^{k} n_l m_l \end{pmatrix}$$

where $I(\theta|\zeta) = \sum_{l=1}^{k} n_l m_l A(x_l) \begin{pmatrix} 1 & x_l \\ x_l & x_l^2 \end{pmatrix}$, $A(x_l) = e^{-b(d-2)x_l}$.

- Asymptotic covariance matrix

$$Cov(\hat{\theta}) = \left(I^*(\theta|\zeta)\right)^{-1}.$$

Properties of MLE

- Invariant property
  If $g(\theta)$ is a function of θ, then the MLE of $g(\theta)$ is $g(\hat{\theta})$.

- δ-method
  The asymptotic variance of $g(\hat{\theta})$ is
  $$AVar\left(g(\hat{\theta})\right) = \nabla g(\theta) Cov(\hat{\theta}) \nabla g(\theta)^T,$$
  where $\nabla g(\theta) = \left(\frac{\partial g(\theta)}{\partial a}, \frac{\partial g(\theta)}{\partial b}, \frac{\partial g(\theta)}{\partial \lambda}\right)$.

The expression of Avar $(\hat{\xi}_q)$

- Asymptotic variance

$$AVar(\hat{\xi}_q|\zeta) = \frac{1}{f_T(\xi_q)^2\Delta t}\left[\frac{h_{1,q}^2 e^{a(d-2)}\sum_{l=1}^{k} A(x_l)x_l^2 m_l n_l}{\lambda\sum_{u<v} A(x_u+x_v)(x_v-x_u)^2 m_u n_u m_v n_v} + \frac{2\lambda^2 h_{2,q}^2 \Delta t}{\sum_{l=1}^{k} m_l n_l}\right],$$

where $h_{1,q} = \frac{\partial F_T(\xi_q|\theta)}{\partial a}, h_{2,q} = \frac{\partial F_T(\xi_q|\theta)}{\partial \lambda}$.

- m_l and n_l are nonidentifiability, because of the assumptions of independent and stationary increment.

The expression of Avar $(\hat{\xi}_q)$

- We use a two-step approach to obtain the optimal design
  1. We first derive an optimal approximate design.
  2. We involve a cost constraint to calculate n_l and m_l.

- Let $N_0 = \sum_{l=1}^{k} m_l n_l$ and $p_{l0} = \frac{n_l m_l}{N_0}$

$$AVar\left(\hat{\xi}_q \middle| \zeta\right) = \frac{1}{f_T\left(\xi_q\right)^2 N_0 \Delta t}\left[\frac{h_{1,q}^2 e^{a(d-2)} \sum_{l=1}^{k} A(x_l) x_l^2 p_{l0}}{\lambda \sum_{u<v}^{k} A(x_u + x_v)(x_v - x_u)^2 p_{u0} p_{v0}} + 2\lambda^2 h_{2,q}^2 \Delta t\right],$$

Optimal approximate design

- Prefix N_0, minimizing Avar $(\hat{\xi}_q)$ is equivalent to minimize

$$G(\zeta) = \frac{\sum_{l=1}^{k} A(x_l) x_l^2 p_{l0}}{\sum_{u<v}^{k} A(x_v + x_u)(x_v - x_u)^2 p_{u0} p_{v0}},$$

$$\zeta = \begin{pmatrix} x_1 & \cdots & x_k \\ p_{10} & \cdots & p_{k0} \end{pmatrix}$$

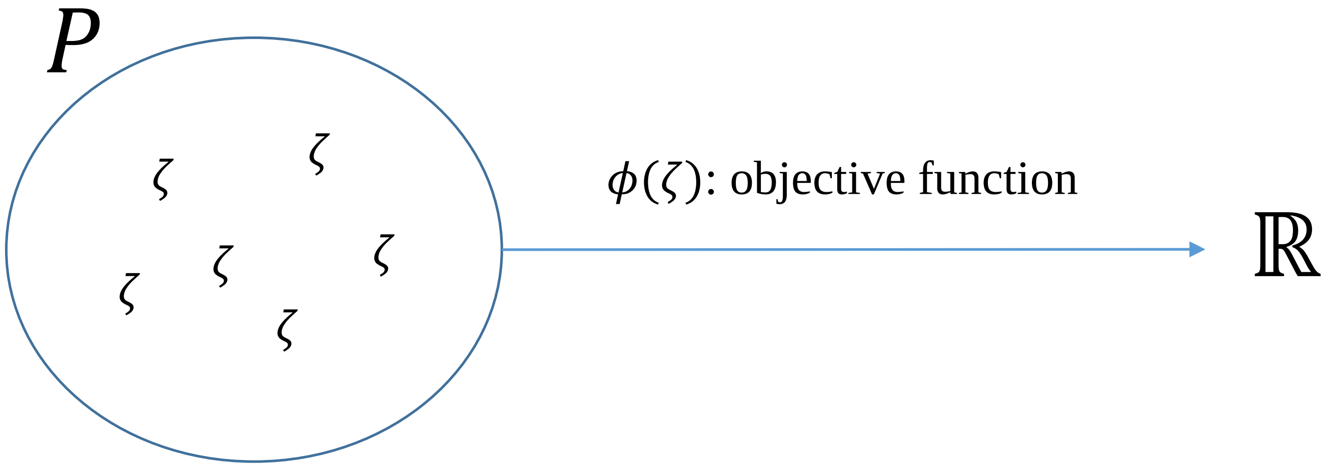
- V-optimal approximate design

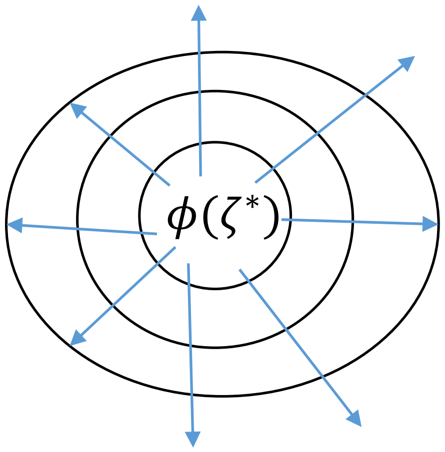
$$\zeta^* = \operatorname{argmin}_{\zeta} G(\zeta)$$

Introduction of general equivalence theorem

$$\zeta = \begin{pmatrix} x_1 & x_2 \\ p_1 & p_2 \end{pmatrix}, \; p_1 + p_2 = 1 \Leftrightarrow \zeta(x) = \begin{cases} p_1, & \text{if } x = x_1 \\ p_2, & \text{if } x = x_2 \end{cases}, \; p_1 + p_2 = 1$$

$$\qquad\qquad\qquad\qquad\qquad\qquad 0, \; \text{o.w.}$$





ζ^* is ϕ-optimal iff the directional derivative of ϕ at ζ^* is zero.

Introduction of general equivalence theorem

$$\zeta = \begin{pmatrix} x_1 & x_2 \\ p_1 & p_2 \end{pmatrix},\ p_1 + p_2 = 1 \Leftrightarrow \zeta(x) = \begin{cases} p_1, & \text{if } x = x_1 \\ p_2, & \text{if } x = x_2 \\ 0, & \text{o.w.} \end{cases},\ p_1 + p_2 = 1$$

Let ζ be the probability measure containing in the set of all probability measure, P, on the design region D. Let ϕ be the function in P and $\eta_1,\ \eta_2 \in P$. Then, the derivative twoard η_2 at a given point η_1 is defined by

$$\Lambda(\eta_1, \eta_2) = \lim_{\varepsilon \downarrow 0} \frac{\phi((1-\varepsilon)\eta_1 + \varepsilon\eta_2) - \phi(\eta_1)}{\varepsilon}.$$

Introduction of general equivalence theorem

$$\zeta^* \text{ is the optimal design iff } \sup_{\eta \in P} \Lambda(\zeta^*, \eta) = 0$$

- Let $\zeta_x(z) = \begin{cases} 1 & \text{if } z = x \\ 0 & o.w. \end{cases}$ be a probability measure with probability 1 at x. Then, $\{\zeta_x | x \in [x_L, 1]\}$ is a basis of P. That is, for any $\eta \in P$, $\eta = \int \zeta_x \, \eta(dx)$.
- For example, if $\eta(z) = \begin{cases} p_1 & \text{if } z = x_1 \\ p_2 & \text{if } z = x_2 \end{cases}$, then

$$\eta = \zeta_{x_1}\eta(x_1) + \zeta_{x_2}\eta(x_2) = \zeta_{x_1}p_1 + \zeta_{x_2}p_2.$$

Introduction of general equivalence theorem

$$\zeta^* \text{ is the optimal design iff } \sup_{\eta\in P}\Lambda(\zeta^*,\eta) = 0$$

- If $\Lambda(\zeta,\eta) = \Lambda\left(\zeta, \zeta_{x_1}p_1 + \zeta_{x_2}p_2\right) = \Lambda\left(\zeta,\zeta_{x_1}\right)p_1 + \Lambda\left(\zeta,\zeta_{x_2}\right)p_2$, then we say $\Lambda(\zeta,\eta)$ is linear in η.

- If $\Lambda(\zeta,\eta)$ is linear in η, then
$$\sup_{\eta\in P}\Lambda(\zeta^*,\eta) = 0 \text{ iff } \sup_{x\in[x_L,1]}\Lambda(\zeta^*,\zeta_x) = 0.$$

Introduction of general equivalence theorem

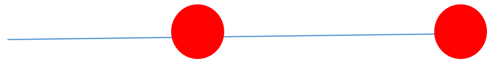
Under some regular conditions, Whittle (1973), Chaloner & Larntz (1989) stated that if ϕ is a concave function, then

$$\zeta^* \text{ is the optimal design iff } \sup_{x \in [x_L, 1]} \Lambda(\zeta^*, x) = 0,$$

where $\Lambda(\zeta^*, x) = \Lambda(\zeta^*, \zeta_x)$

Conjecture V-optimal design

1. $d < 2, \zeta^{\Delta} = \begin{pmatrix} \max(x_L, \rho_1) & 1 \\ p_{10}^{\Delta} & p_{20}^{\Delta} \end{pmatrix}$



2. $d = 2, \zeta^{\Delta} = \begin{pmatrix} x_L & 1 \\ p_{10}^{\Delta} & p_{20}^{\Delta} \end{pmatrix}$

3. $d > 2, \zeta^{\Delta} = \begin{pmatrix} x_L & \min(1, \rho_2) \\ p_{10}^{\Delta} & p_{20}^{\Delta} \end{pmatrix}$

where $\rho_1 = 1 + [1 + W(e^{-1})]\frac{2}{b(d-2)}$, $\rho_2 = x_L + [1 + W(e^{-1})]\frac{2}{b(d-2)}$, and $p_{10}^{\Delta} = \frac{x_2^{\Delta} A(x_2^{\Delta}/2)}{x_1^{\Delta} A(x_1^{\Delta}/2) + x_2^{\Delta} A(x_2^{\Delta}/2)}$, $p_{20}^{\Delta} = 1 - p_{10}^{\Delta}$.

Conjecture V-optimal design

Theorem

$\sup_x \Lambda(\zeta^{\Delta}, x) = 0,$ and hence ζ^{Δ} is the global V-optimal design.

Tung, H. P., Lee, I. C., & Tseng, S. T. (2022). Analytical approach for designing accelerated degradation tests under an exponential dispersion model. *Journal of Statistical Planning and Inference*, *218*, 73-84.

Optimal allocation rule based on cost constraint

- Asymptotic variance

$$AVar\left(\hat{\xi}_q|\zeta\right) = \frac{1}{f_T\left(\xi_q\right)^2 \Delta t}\left[\frac{h_{1,q}^2 e^{a(d-2)} \sum_{l=1}^{k} A(x_l) x_l^2 m_l n_l}{\lambda \sum_{u<v}^{k} A(x_u + x_v)(x_v - x_u)^2 m_u n_u m_v n_v} + \frac{2\lambda^2 h_{2,q}^2 \Delta t}{\sum_{l=1}^{k} m_l n_l}\right],$$

where $h_{1,q} = \frac{\partial F_T(\xi_q|\theta)}{\partial a}, h_{2,q} = \frac{\partial F_T(\xi_q|\theta)}{\partial \lambda}$.

- m_l and n_l are nonidentifiability, because of the assumptions of independent and stationary increment.

Optimal allocation rule based on cost constraint

- Cost Constraint

$$C(m_1, m_2, n_1, n_2) = c_{op}\Delta t(m_1 + m_2) + c_{it}(n_1 + n_2)$$

- Minimize $C(m_1, m_2, n_1, n_2)$

  subjected to $m_1 n_1 = p_1^{\Delta} N_0$ and $m_2 n_2 = p_2^{\Delta} N_0$

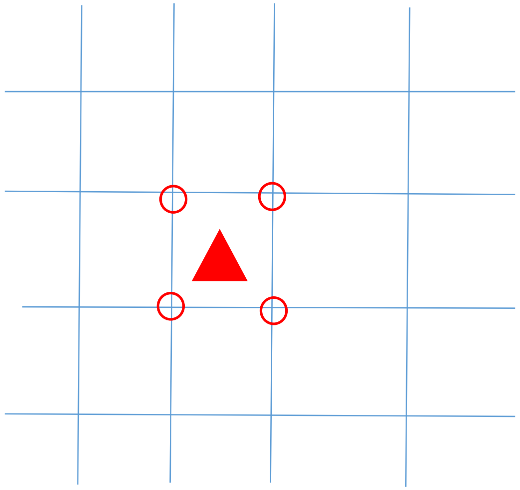
Theorem

$$m_1^{\Delta} = \sqrt{\frac{c_{it} p_{10}^{\Delta} N_0}{c_{op}\Delta t}},\ m_2^{\Delta} = \sqrt{\frac{c_{it} p_{20}^{\Delta} N_0}{c_{op}\Delta t}},\ n_1^{\Delta} = \sqrt{\frac{c_{op}\Delta t p_{10}^{\Delta} N_0}{c_{it}}},\ n_2^{\Delta} = \sqrt{\frac{c_{op}\Delta t p_{20}^{\Delta} N_0}{c_{it}}}.$$

Optimal allocation rule based on cost constraint

$$(m_1^*, m_2^*, n_1^*, n_2^*) = \underset{(m_1,m_2,n_1,n_2)\in\Omega}{\arg\min} \mathrm{AVar}(\hat{\xi}_q|\zeta)$$

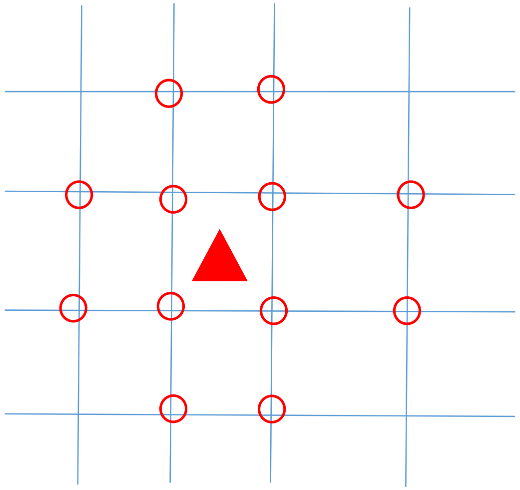
$$\Omega = \{(m_1, m_2, n_1, n_2) \in \mathbb{N}^4 |\ |m_l - \lfloor m_l^{\Delta} \rfloor| \leq \nu, |n_l - \lfloor n_l^{\Delta} \rfloor| \leq \nu \text{ for } l = 1, 2\}$$



Optimal allocation rule based on cost constraint

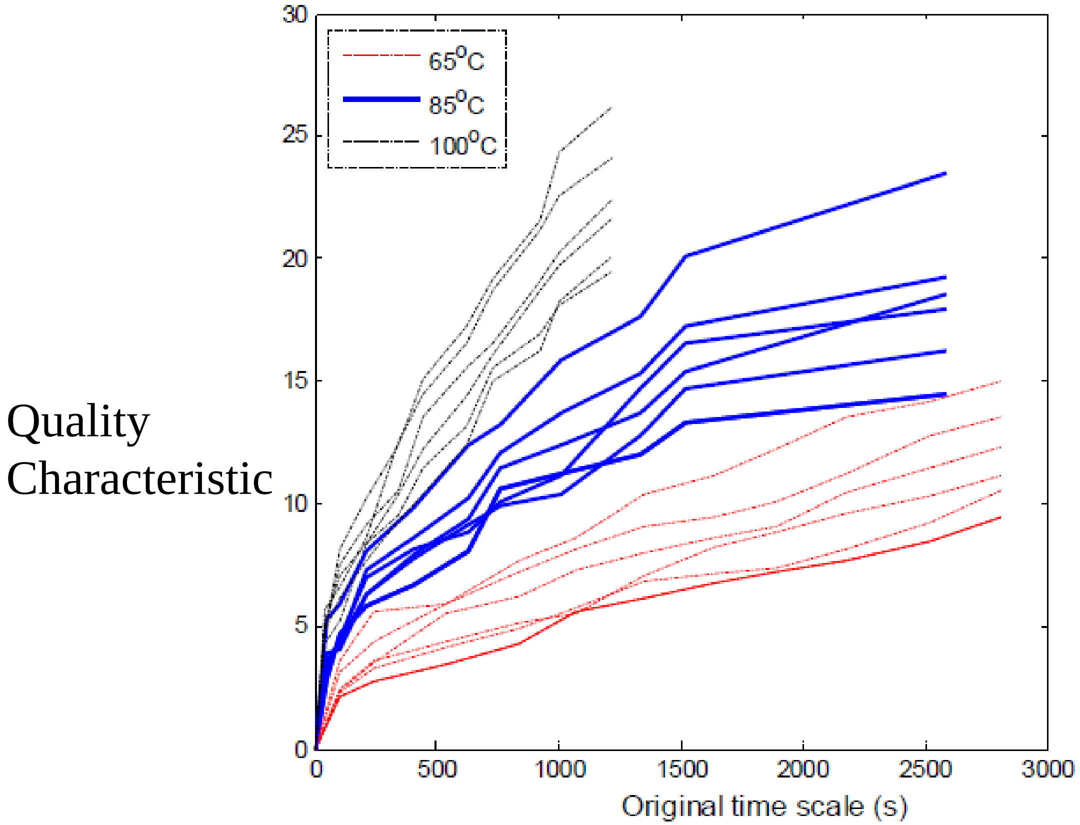
$$(m_1^*, m_2^*, n_1^*, n_2^*) = \underset{(m_1, m_2, n_1, n_2) \in \Omega}{\arg\min} \; \mathrm{AVar}(\hat{\xi}_q | \zeta)$$

$$\Omega = \{(m_1, m_2, n_1, n_2) \in \mathbb{N}^4 | \; |m_l - \lfloor m_l^\Delta \rfloor| \le \nu, |n_l - \lfloor n_l^\Delta \rfloor| \le \nu \text{ for } l = 1, 2\}$$



An illustrative example

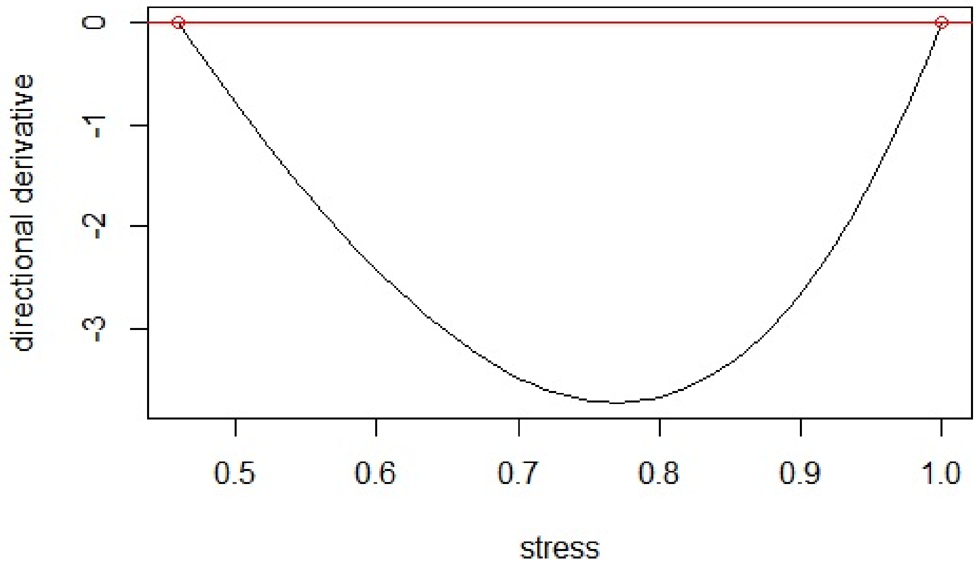
- Stress relaxation data (Yang, 2007)
- Stress levels: 65°C, 85°C, 100°C
- Normal use condition: 40°C
- $(d, a, b, \lambda) = (1.4, 1.95, 1.83, 2.20)$
- Standardized design region [0.46,1]



An illustrative example

- Based on the V-optimal design Theorem, we have

$$\zeta^{\Delta} = \begin{pmatrix} x_1^{\Delta} & x_2^{\Delta} \\ p_{10}^{\Delta} & p_{20}^{\Delta} \end{pmatrix} = \begin{pmatrix} 0.46 & 1 \\ 0.745 & 0.255 \end{pmatrix}$$



An illustrative example

- Assuming $N_0 = 1000, c_{op} = 1.9, c_{it} = 53, \Delta t = 4$

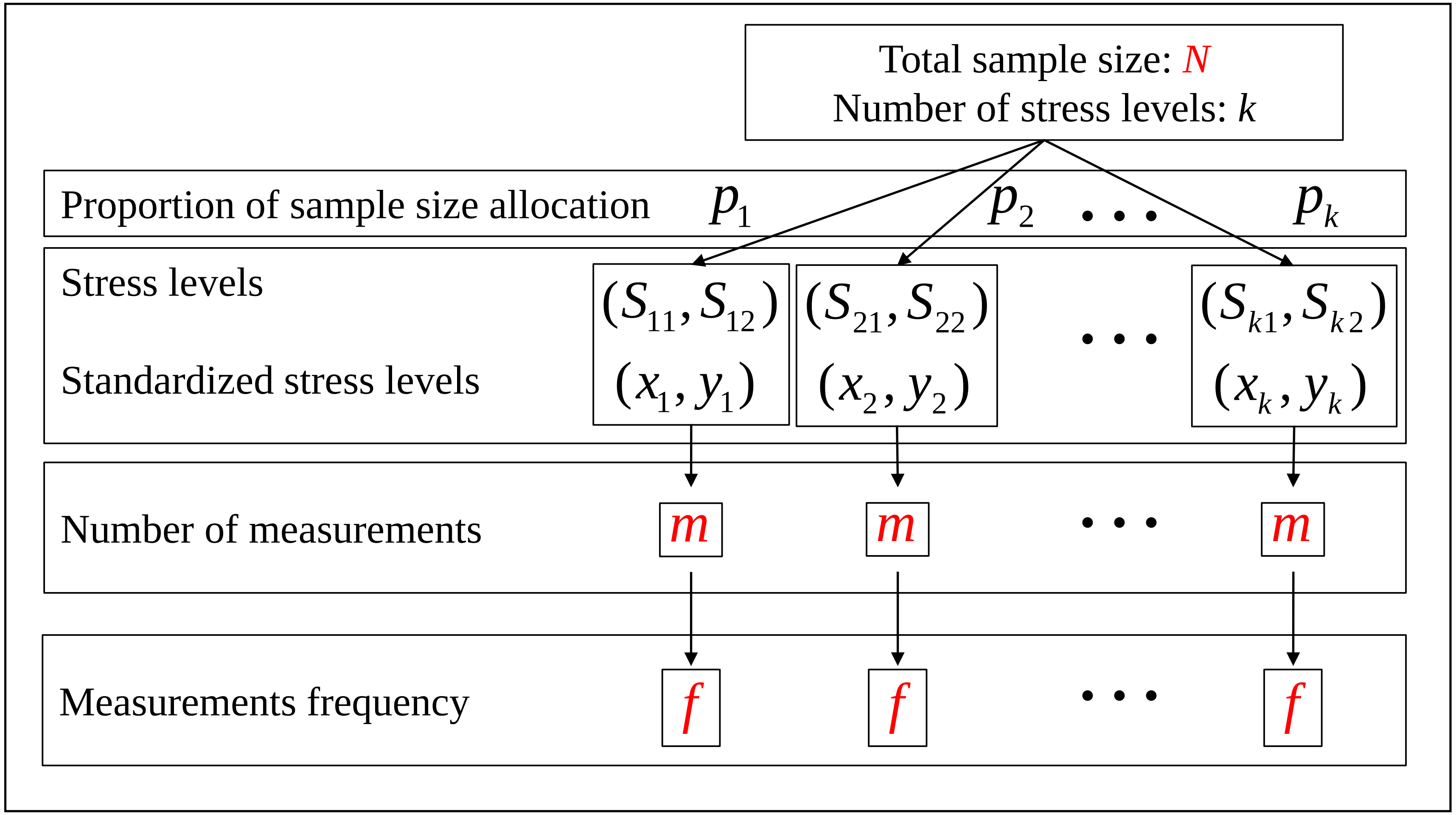
$$\zeta^{\Delta} = \begin{pmatrix} x_1^{\Delta} & x_2^{\Delta} \\ n_1^{\Delta} & n_2^{\Delta} \\ m_1^{\Delta} & m_2^{\Delta} \end{pmatrix} = \begin{pmatrix} 0.46 & 1 \\ 10.34 & 6.05 \\ 72.09 & 42.16 \end{pmatrix}$$

An illustrative example

ν	m_1	m_2	n_1	n_2	p_{10}	p_{20}	AVar	N_0	time (sec)
1	73	43	10	6	0.74	0.26	51.73	988	0.01
2	74	42	10	6	0.75	0.25	51.51	992	0.06
3	69	40	11	6	0.76	0.24	51.21	999	0.09
4	76	40	10	6	0.76	0.24	51.16	1000	0.50
5	76	40	10	6	0.76	0.24	51.16	1000	1.10
Grid search	76	40	10	6	0.76	0.24	51.16	1000	224.11

- Optimal design for EDADTs of two accelerating variables without interaction
  - Problem formulation
  - Degenerate design
  - Non-degenerate design
  - An illustrate example

Problem formulation



$$\zeta = \begin{pmatrix} (x_1, y_1) & \cdots & (x_k, y_k) \\ p_1 & \cdots & p_k \end{pmatrix}$$

Problem formulation

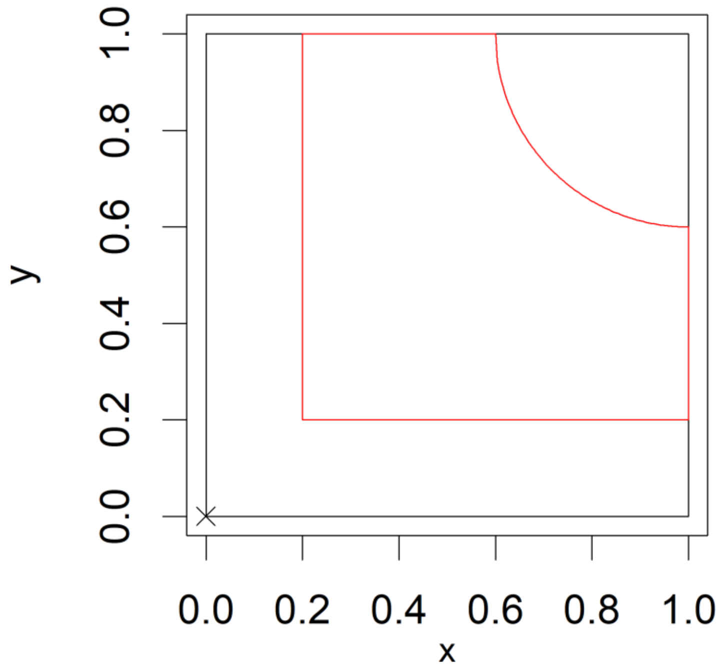
- Let $Z_i(t_j|x_l)$ $(i=1,\ldots,n_i,\ j=1,\ldots m,\ l=1,\ldots,k)$ denote the degradation of ith test unit at time $t_j = j\times\Delta t$ under lth stress-level (x_l, y_l).

- $Z_i(t_j|x_l) \sim ED(\mu(x_l,y_l)t_j,\lambda)$, $\ln(\mu(x_l,y_l)) = \alpha_0+\alpha_1 x_l+\alpha_2 y_l, \alpha_1, \alpha_2>0, (x_l,y_l)\in D_2$
  $\Delta Z_{ijl} = Z_i(t_j|x_l) - Z_i(t_{(j-1)}|x_l)$ has the probability density function:

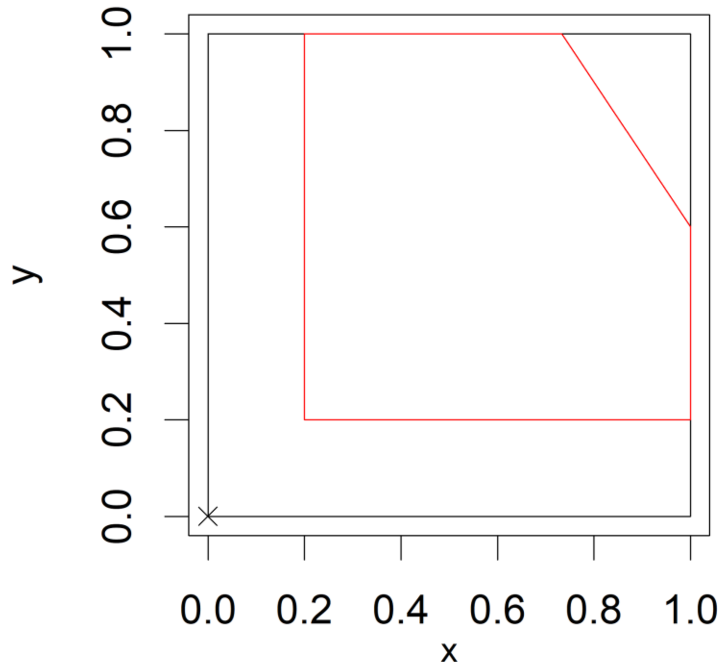
$$f(\Delta z_{ijl} \mid \mu(x_l,y_l),\lambda) = c(\Delta z_{ijl} \mid \lambda, \Delta t)e^{\lambda\{\varpi(\mu(x_l,y_l))\Delta z_{ijl} - \Delta t\kappa[\varpi(\mu(x_l,y_l))]\}}$$

Problem formulation

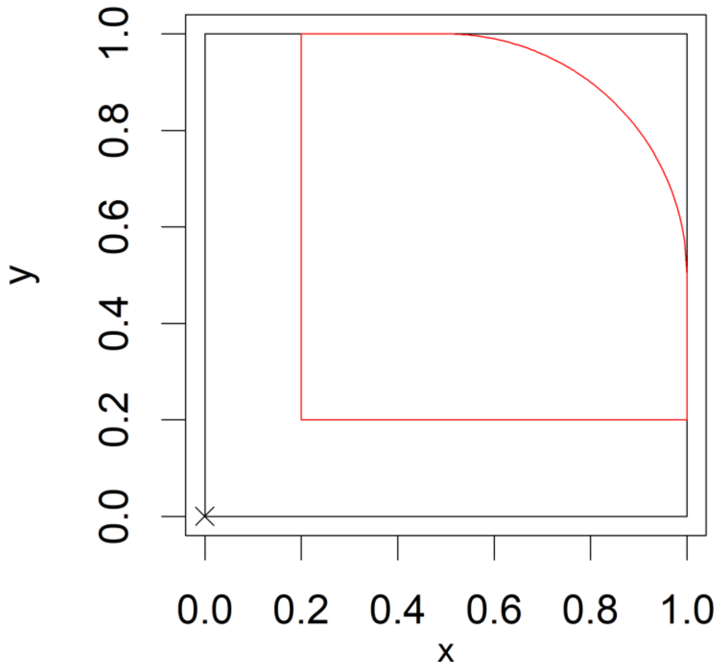
D_2 can be



(a)



(b)

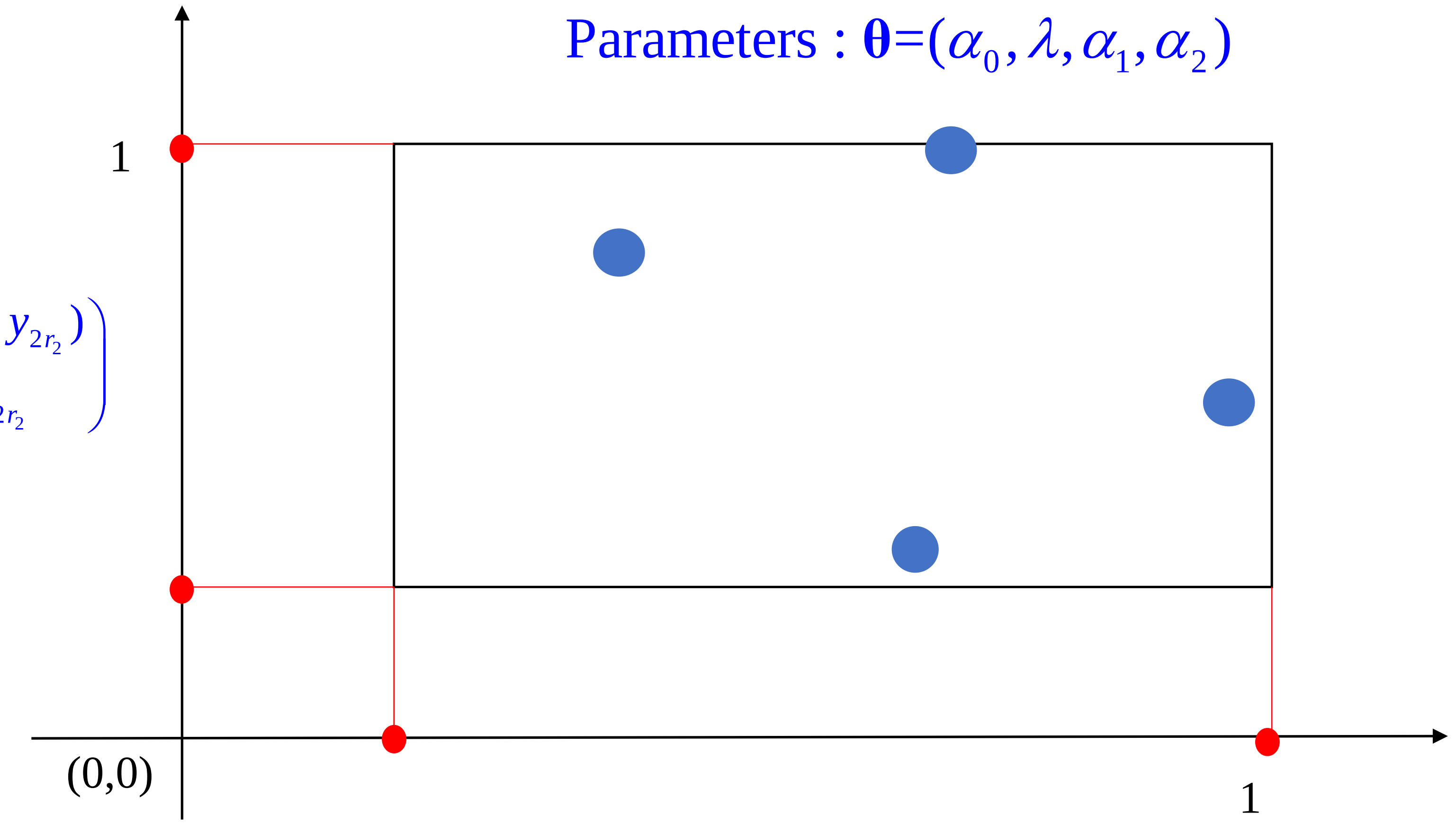


(c)

Escobar and Meeker (1995)

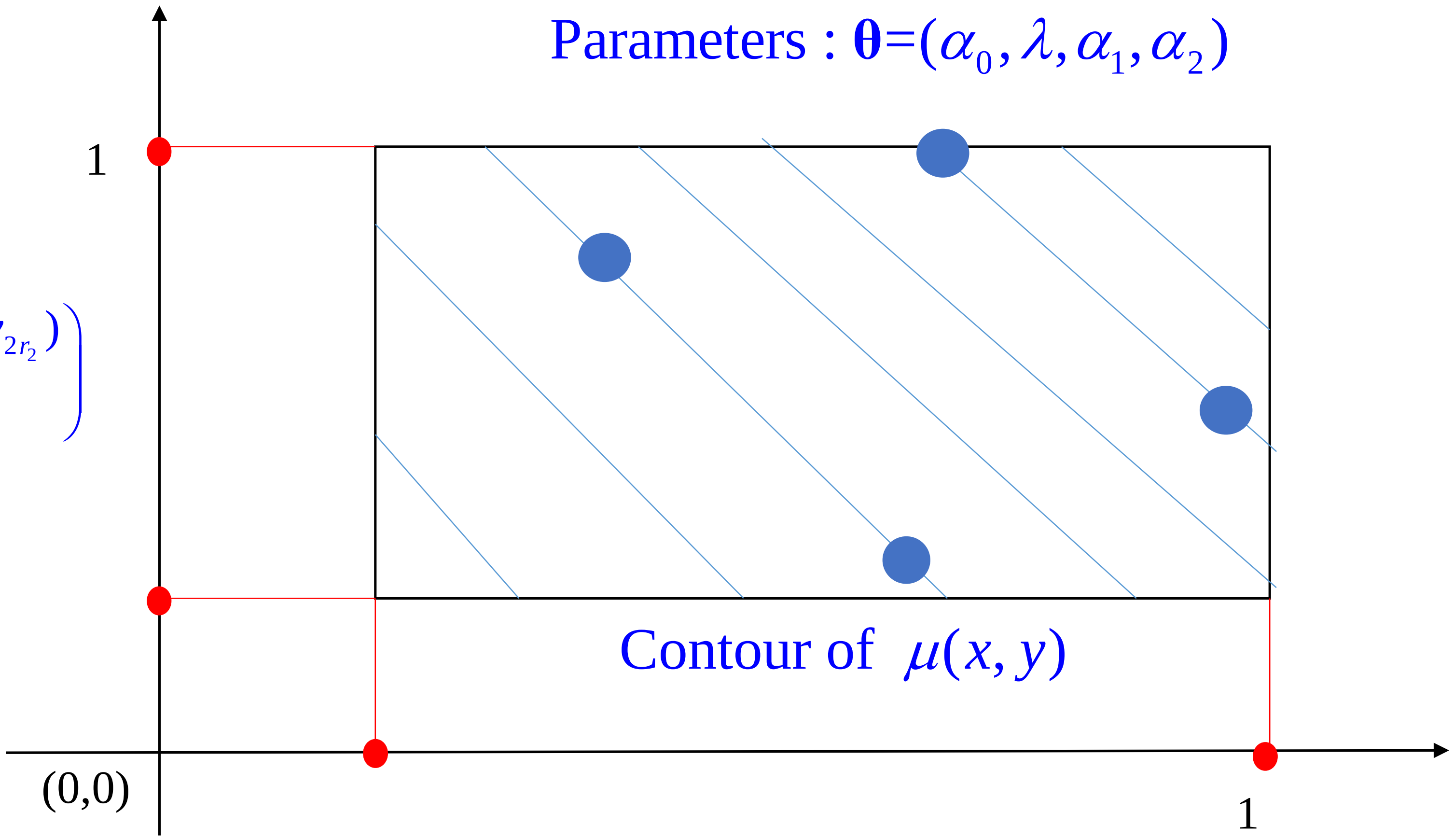
Parameters : $\boldsymbol{\theta}=(\alpha_0, \lambda, \alpha_1, \alpha_2)$

$$\begin{pmatrix} (x_{11}, y_{11}) & \cdots & (x_{1r_1}, y_{1r_1}) & (x_{21}, y_{21}) & \cdots & (x_{2r_2}, y_{2r_2}) \\ p_{11} & \cdots & p_{1r_1} & p_{21} & \cdots & p_{2r_2} \end{pmatrix}$$



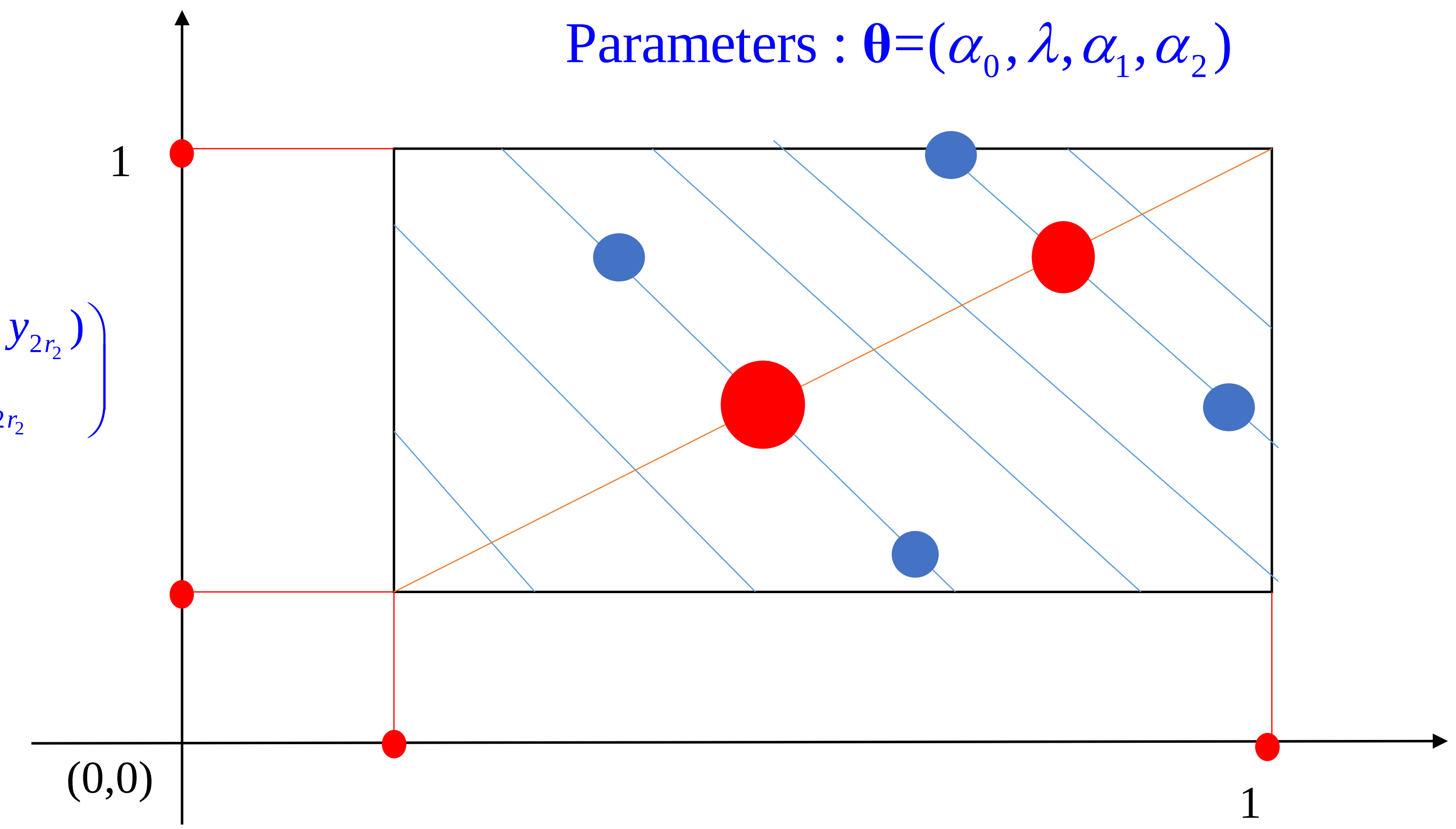
Escobar and Meeker (1995)

$$\begin{pmatrix} (x_{11}, y_{11}) & \cdots & (x_{1r_1}, y_{1r_1}) & (x_{21}, y_{21}) & \cdots & (x_{2r_2}, y_{2r_2}) \\ p_{11} & \cdots & p_{1r_1} & p_{21} & \cdots & p_{2r_2} \end{pmatrix}$$



Escobar and Meeker (1995)

$$\begin{pmatrix} (x_{11}, y_{11}) & \cdots & (x_{1r_1}, y_{1r_1}) & (x_{21}, y_{21}) & \cdots & (x_{2r_2}, y_{2r_2}) \\ p_{11} & \cdots & p_{1r_1} & p_{21} & \cdots & p_{2r_2} \end{pmatrix}$$



Escobar and Meeker (1995)

$$\begin{pmatrix} (x_1, x_1) & (x_2, x_2) \\ p_1 & p_2 \end{pmatrix}$$

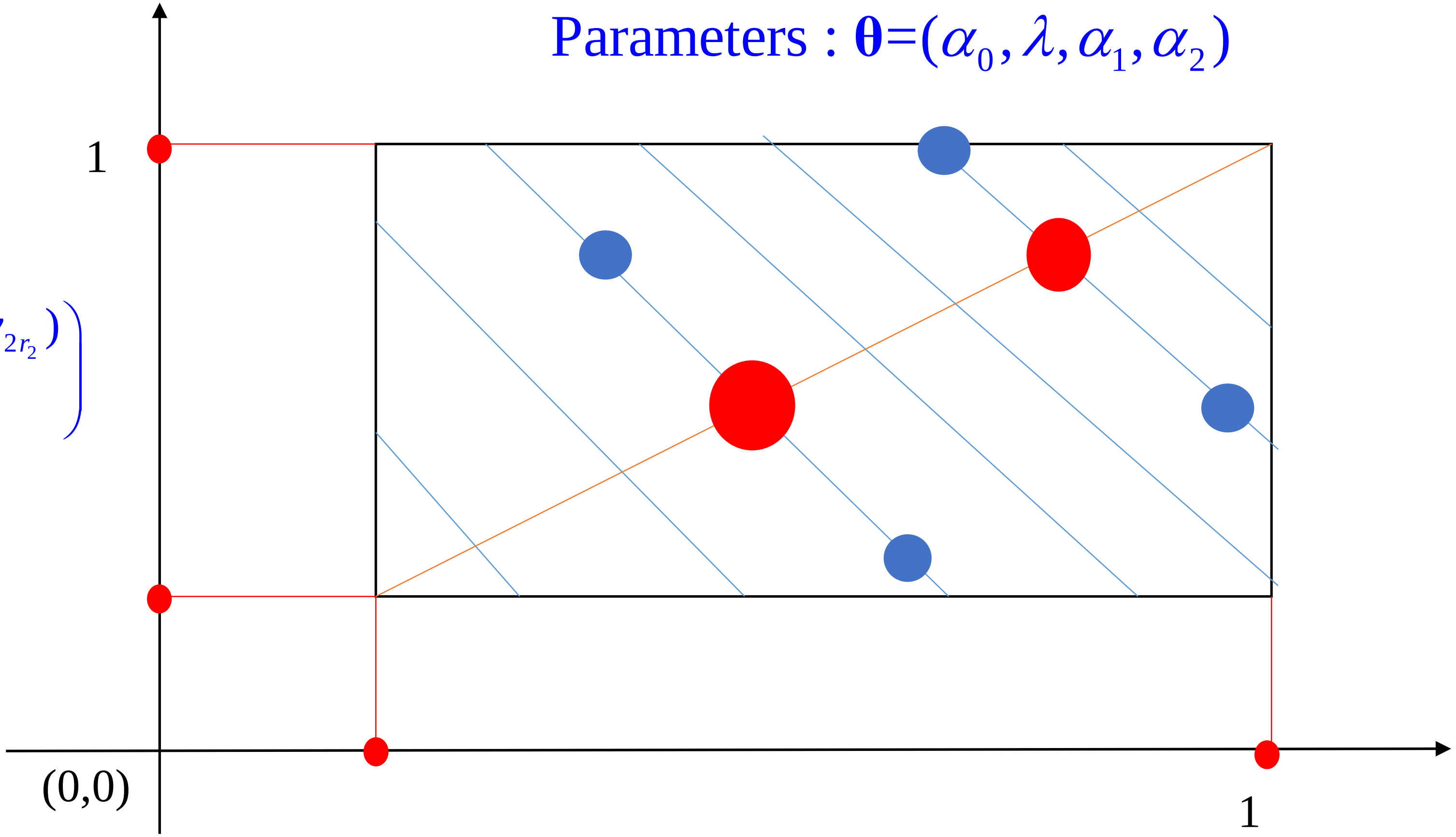
$$\begin{pmatrix} (x_{11}, y_{11}) & \cdots & (x_{1r_1}, y_{1r_1}) & (x_{21}, y_{21}) & \cdots & (x_{2r_2}, y_{2r_2}) \\ p_{11} & \cdots & p_{1r_1} & p_{21} & \cdots & p_{2r_2} \end{pmatrix}$$

$$\sum_{r=1}^{r_l} p_{lr} = p_l$$

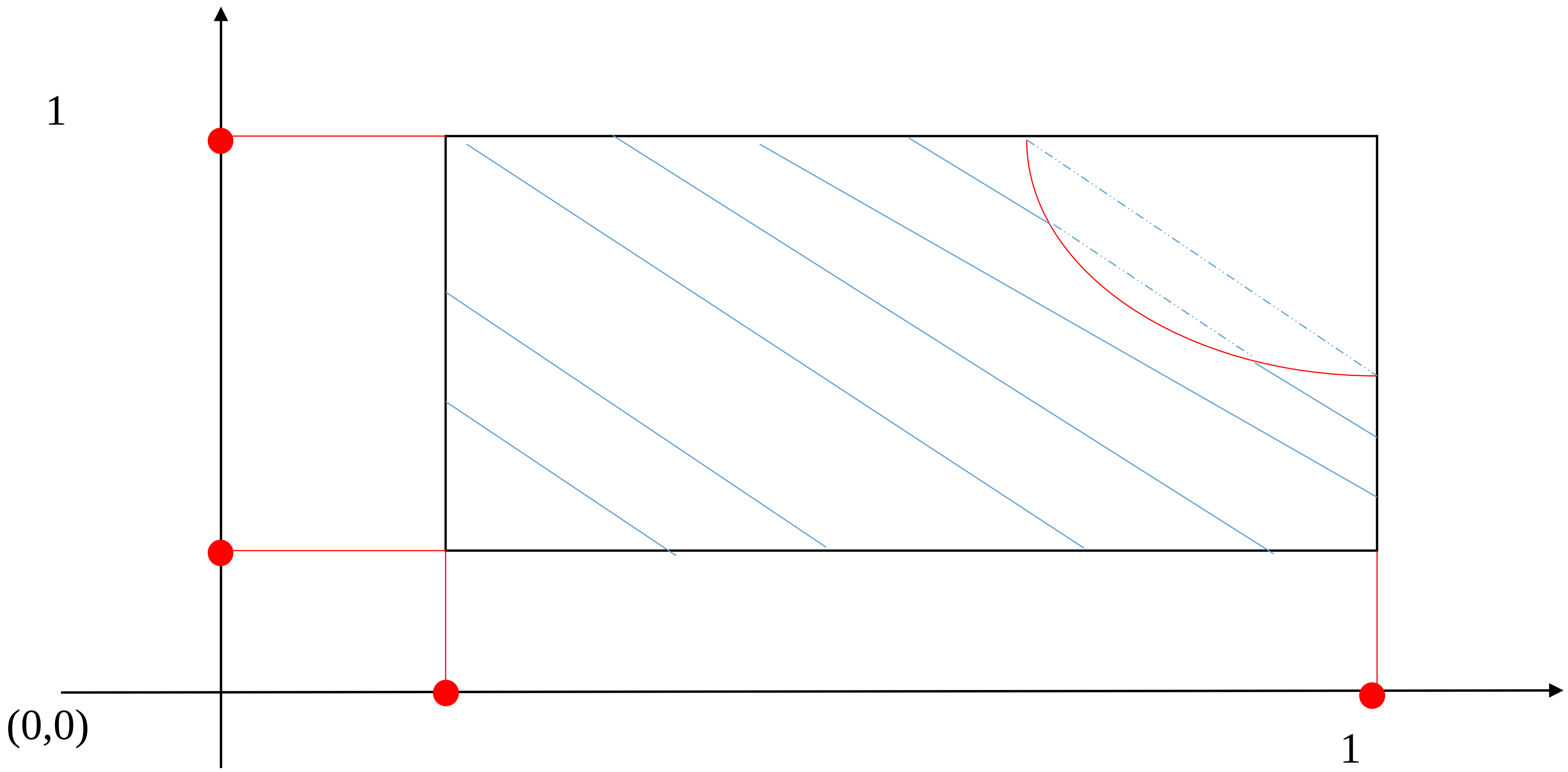
$$\alpha_1 x_{lr} + \alpha_2 y_{lr} = \alpha_1 x_l + \alpha_2 x_l$$

Parameters : $\boldsymbol{\theta}' = (\alpha_0, \lambda, \alpha_1')$

Parameters : $\boldsymbol{\theta} = (\alpha_0, \lambda, \alpha_1, \alpha_2)$

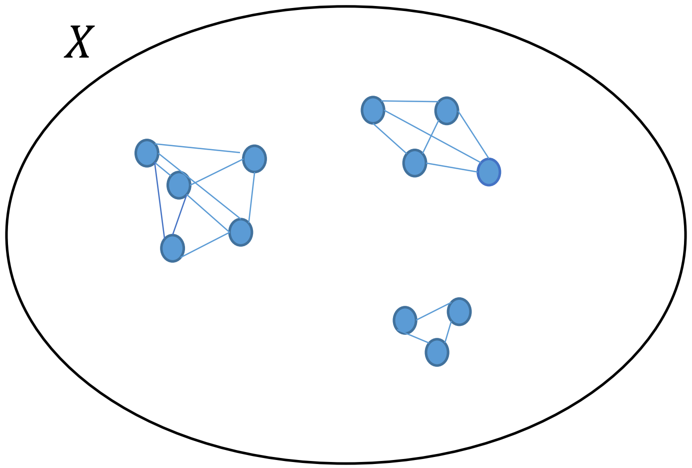


Irregular design region



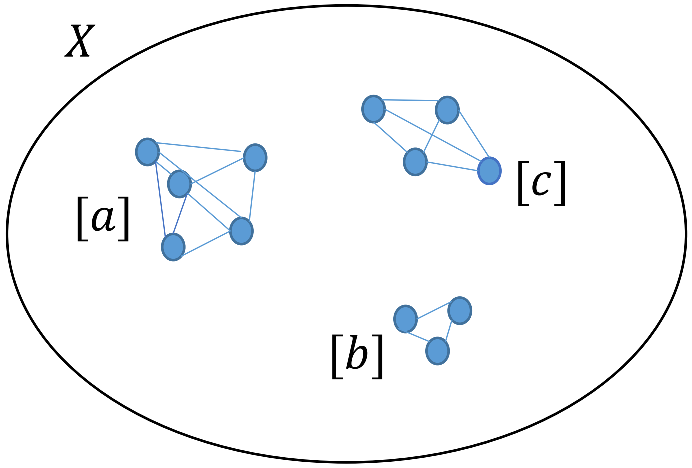
Equivalence relation

- Binary relation: Given sets X and Y, a binary relation R over sets X and Y is a subset of $X \times Y$.
- If $(x, y) \in R$, then we say x is related to y and is denoted by xRy or $x \sim y$.
- Equivalence relation: A binary relation ~ over sets X and X and satisfies the following properties: for $a, b, c \in X$,
  - $a \sim a$ (reflexivity)
  - If $a \sim b$ then $b \sim a$ (symmetry)
  - If $a \sim b$ and $b \sim c$ then $a \sim c$ (transitivity)



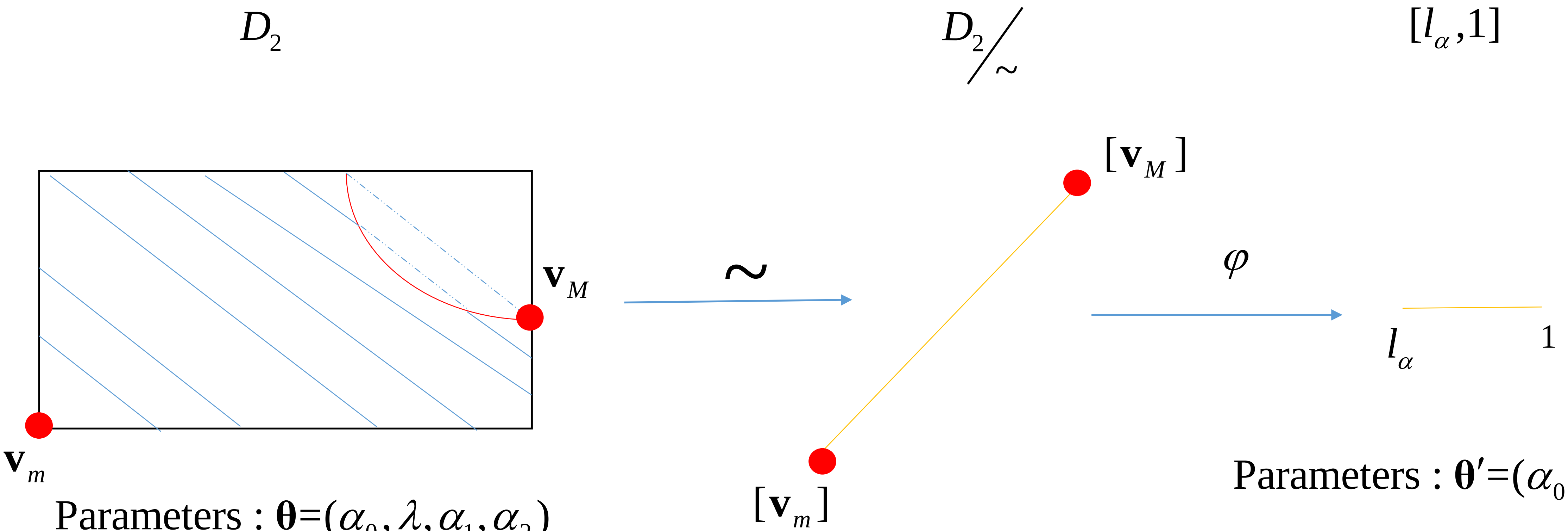
Equivalence relation

- Equivalence class: $[a] = \{x \in X \mid x \sim a\}$
- Quotient set: $X/\sim\; = \{[x] \mid x \in X\}$



$X/\sim\; = \{[a], [b], [c]\}$

Non-degenerate design → Degenerate design



Parameters : $\boldsymbol{\theta} = (\alpha_0, \lambda, \alpha_1, \alpha_2)$

Parameters : $\boldsymbol{\theta}' = (\alpha_0, \lambda, \alpha_1')$

$\alpha_1' = \boldsymbol{\alpha}^T \mathbf{v}_M$

Find degenerate designs

- Let $\sim$ be a relation on D_2 and for $\mathbf{v}_1, \mathbf{v}_2 \in D_2$, $\mathbf{v}_1 \sim \mathbf{v}_2$ if $\boldsymbol{\alpha}^T \mathbf{v}_1 = \boldsymbol{\alpha}^T \mathbf{v}_2$ where $\boldsymbol{\alpha} = (\alpha_1, \alpha_2)^T$.
  $\sim$ is an equivalence relation.

- The equivalence class of an element $\mathbf{v}_1$ in D_2, denoted by $[\mathbf{v}_1]$ is the set $\{\mathbf{v} \in D_2 | \mathbf{v} \sim \mathbf{v}_1\}$.
  The quotient set of D_2 is denoted by $D_2/\sim = \{[\mathbf{v}] \mid \mathbf{v} \in D_2\}$.

- Let $\mathbf{v}_m, \mathbf{v}_M \in D_2$ be the points that minimize and maximize $\mu(\mathbf{v})$, respectively.
  Let φ be a map from $D_2/\sim$ to $[l_\alpha, 1]$, where $l_\alpha = \boldsymbol{\alpha}^T \mathbf{v}_m / \boldsymbol{\alpha}^T \mathbf{v}_M$, and is defined by

$$\varphi([\mathbf{v}]) = \boldsymbol{\alpha}^T \mathbf{v} / \boldsymbol{\alpha}^T \mathbf{v}_M .$$

Degenerate design

- Let ζ^d be the degenerate design on $[l_\alpha, 1]$ with stress levels $u_1 < \cdots < u_k$.

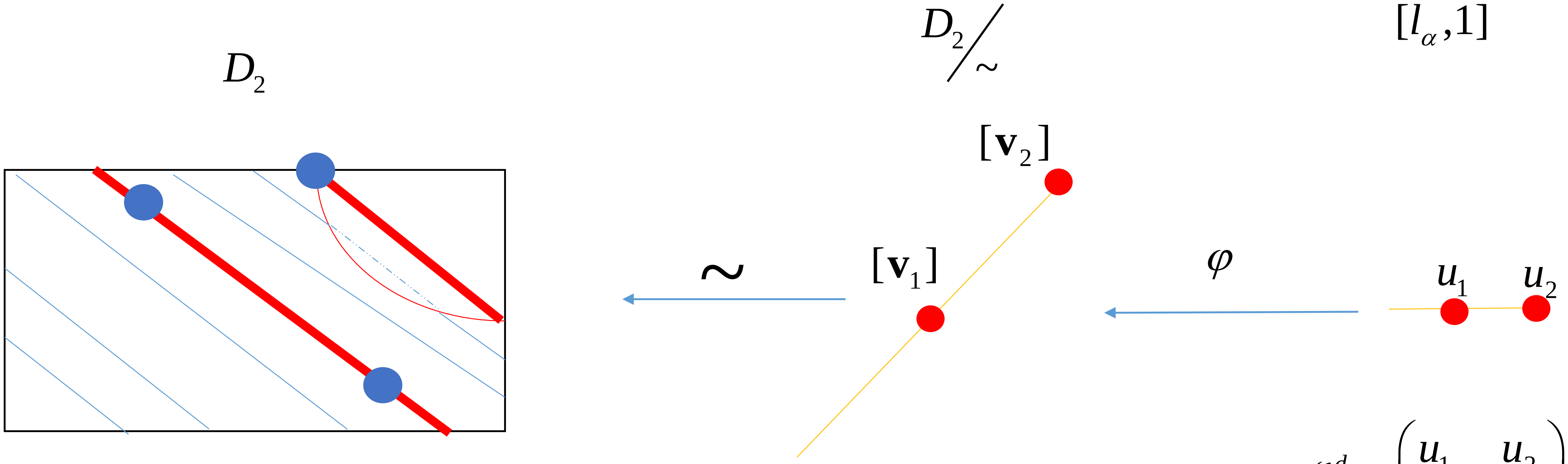
$$\zeta^d = \begin{pmatrix} u_1 & \cdots & u_k \\ p_1 & \cdots & p_k \end{pmatrix}$$

- To minimize $\mathrm{AVar}(\hat{\xi}_q | \zeta^d)$ is equivalent to minimize $G^d(\zeta^d)$.

$$G^d(\zeta^d) = [1\ 0][F_{11}^d - F_{12}^d (F_{22}^d)^{-1} (F_{12}^d)^{\mathrm{T}}]^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$F^d = N\lambda m\Delta t \left[\begin{array}{cc|c} \sum_{l=1}^k A_l p_l & 0 & \sum_{l=1}^k A_l p_l u_l \\ 0 & \frac{1}{2\lambda^3 \Delta t} & 0 \\ \hline \sum_{l=1}^k A_l p_l u_l & 0 & \sum_{l=1}^k A_l p_l u_l^2 \end{array}\right] = N\lambda m\Delta t \begin{bmatrix} F_{11}^d & F_{12}^d \\ (F_{12}^d)^T & F_{22}^d \end{bmatrix}$$

Degenerate design → Non-degenerate design



$$\zeta^o = \begin{pmatrix} \mathbf{v}_{11} & \mathbf{v}_{12} & \mathbf{v}_{21} \\ p_{11} & p_{12} & p_{21} \end{pmatrix}$$

$$\zeta^d = \begin{pmatrix} u_1 & u_2 \\ p_1 & p_2 \end{pmatrix}$$

Non-degenerate design

- Asymptotic variance

$$\mathrm{AVar}(\hat{\xi}_q|\zeta^o) = \frac{1}{f_T(\xi_q)^2}\left[\frac{\partial F_T(\xi_q|\boldsymbol{\theta})}{\partial \alpha_0}\ \frac{\partial F_T(\xi_q|\boldsymbol{\theta})}{\partial \lambda}\ 0\ 0\right](F^o)^{-1}\left[\frac{\partial F_T(\xi_q|\boldsymbol{\theta})}{\partial \alpha_0}\ \frac{\partial F_T(\xi_q|\boldsymbol{\theta})}{\partial \lambda}\ 0\ 0\right]^{\mathrm{T}}$$

- To minimize AVar($\hat{\xi}_q|\zeta^o$) is equivalent to minimize $G^o(\zeta^o)$

$$G^o(\zeta^o) = [1\ 0][F^o_{11} - F^o_{12}(F^o_{22})^{-1}(F^o_{12})^{\mathrm{T}}]^{-1}\begin{bmatrix}1\\0\end{bmatrix}.$$

$$F^o = N\lambda m\Delta t\left[\begin{array}{cc|cc} \sum_{l=1}^{k_0} A_l p_l & 0 & \sum_{l=1}^{k_0} A_l \sum_{r=1}^{r_l} p_{lr}x_{lr} & \sum_{l=1}^{k_0} A_l \sum_{r=1}^{r_l} p_{lr}y_{lr} \\ 0 & \frac{1}{2\lambda^3\Delta t} & 0 & 0 \\ \hline \sum_{l=1}^{k_0} A_l \sum_{r=1}^{r_l} p_{lr}x_{lr} & 0 & \sum_{l=1}^{k_0} A_l \sum_{r=1}^{r_l} p_{lr}x_{lr}^2 & \sum_{l=1}^{k_0} A_l \sum_{r=1}^{r_l} p_{lr}x_{lr}y_{lr} \\ \sum_{l=1}^{k_0} A_l \sum_{r=1}^{r_l} p_{lr}y_{lr} & 0 & \sum_{l=1}^{k_0} A_l \sum_{r=1}^{r_l} p_{lr}x_{lr}y_{lr} & \sum_{l=1}^{k_0} A_l \sum_{r=1}^{r_l} p_{lr}y_{lr}^2 \end{array}\right]$$

$$= N\lambda m\Delta t\begin{bmatrix} F^o_{11} & F^o_{12} \\ (F^o_{12})^T & F^o_{22}\end{bmatrix},$$

Degenerate and non-degenerate design

$$G^d(\zeta^d) = [1\ 0][F_{11}^d - F_{12}^d (F_{22}^d)^{-1} (F_{12}^d)^{\mathrm{T}}]^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$F^d = N\lambda m\Delta t \left[\begin{array}{cc|c} \sum_{l=1}^{k} A_l p_l & 0 & \sum_{l=1}^{k} A_l p_l u_l \\ 0 & \frac{1}{2\lambda^3\Delta t} & 0 \\ \hline \sum_{l=1}^{k} A_l p_l u_l & 0 & \sum_{l=1}^{k} A_l p_l u_l^2 \end{array}\right]$$

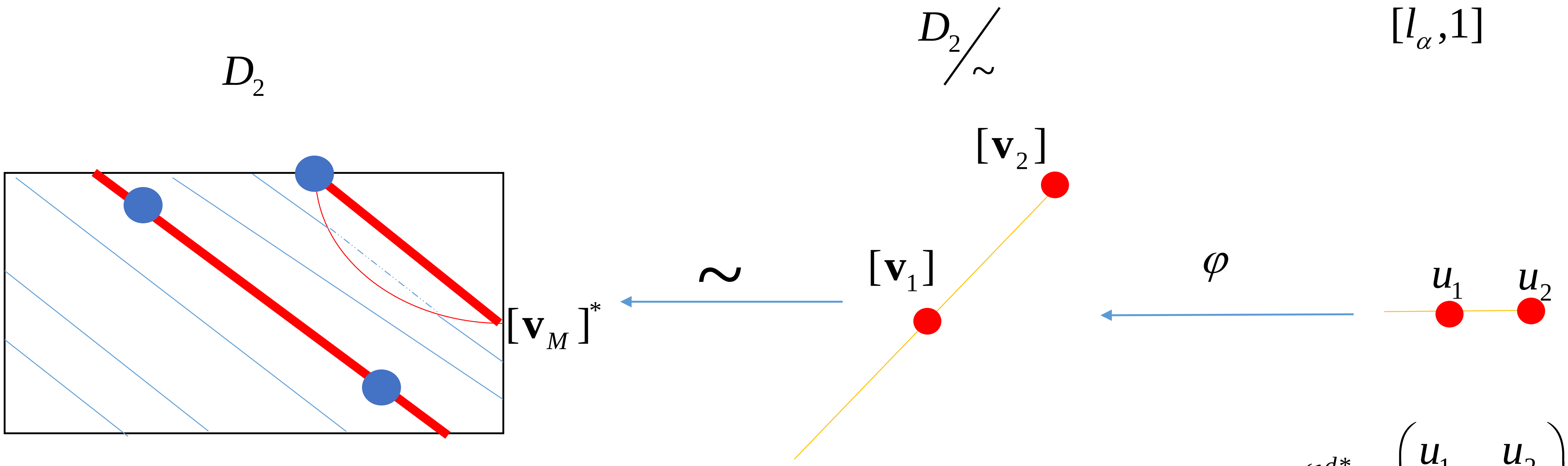
$$= N\lambda m\Delta t \begin{bmatrix} F_{11}^d & F_{12}^d \\ (F_{12}^d)^T & F_{22}^d \end{bmatrix}$$

$$G^o(\zeta^o) = [1\ 0][F_{11}^o - F_{12}^o (F_{22}^o)^{-1} (F_{12}^o)^{\mathrm{T}}]^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

$$F^o = N\lambda m\Delta t \left[\begin{array}{cc|cc} \sum_{l=1}^{k_0} A_l p_l & 0 & \sum_{l=1}^{k_0} A_l \sum_{r=1}^{r_l} p_{lr} x_{lr} & \sum_{l=1}^{k_0} A_l \sum_{r=1}^{r_l} p_{lr} y_{lr} \\ 0 & \frac{1}{2\lambda^3\Delta t} & 0 & 0 \\ \hline \sum_{l=1}^{k_0} A_l \sum_{r=1}^{r_l} p_{lr} x_{lr} & 0 & \sum_{l=1}^{k_0} A_l \sum_{r=1}^{r_l} p_{lr} x_{lr}^2 & \sum_{l=1}^{k_0} A_l \sum_{r=1}^{r_l} p_{lr} x_{lr} y_{lr} \\ \sum_{l=1}^{k_0} A_l \sum_{r=1}^{r_l} p_{lr} y_{lr} & 0 & \sum_{l=1}^{k_0} A_l \sum_{r=1}^{r_l} p_{lr} x_{lr} y_{lr} & \sum_{l=1}^{k_0} A_l \sum_{r=1}^{r_l} p_{lr} y_{lr}^2 \end{array}\right]$$

$$= N\lambda m\Delta t \begin{bmatrix} F_{11}^o & F_{12}^o \\ (F_{12}^o)^T & F_{22}^o \end{bmatrix},$$

Theorem 1

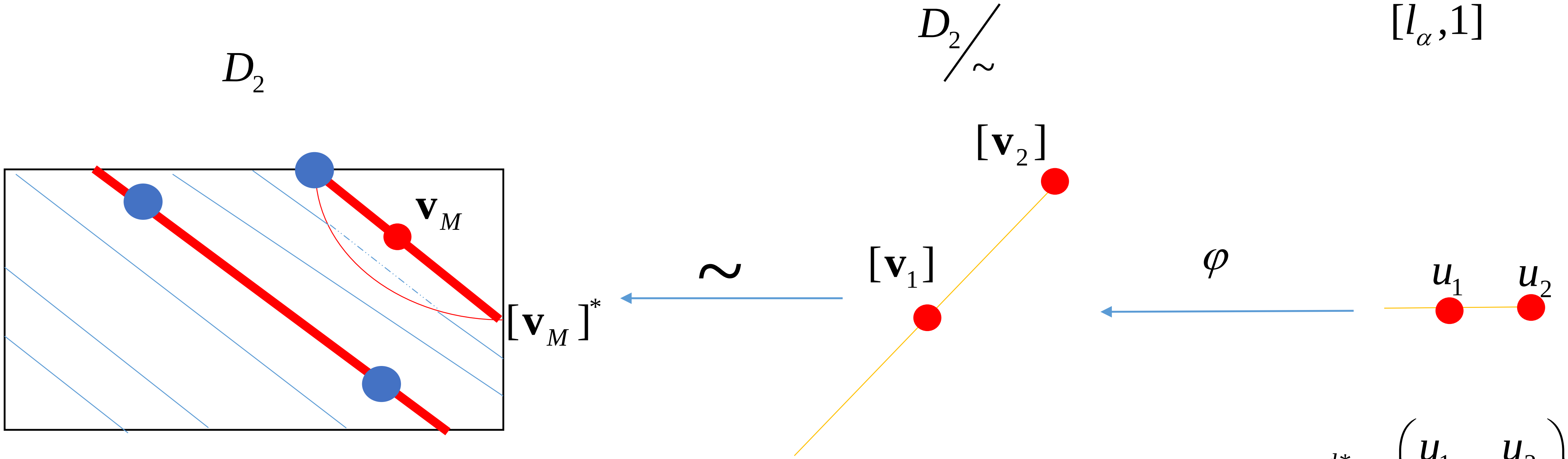


$$\zeta^o = \begin{pmatrix} \mathbf{v}_{11} & \mathbf{v}_{12} & \mathbf{v}_{21} \\ p_{11} & p_{12} & p_{21} \end{pmatrix}$$

$$G^o(\zeta^o) \geq G^d(\zeta^d)$$

$$\zeta^{d*} = \begin{pmatrix} u_1 & u_2 \\ p_1 & p_2 \end{pmatrix}$$

Theorem 1



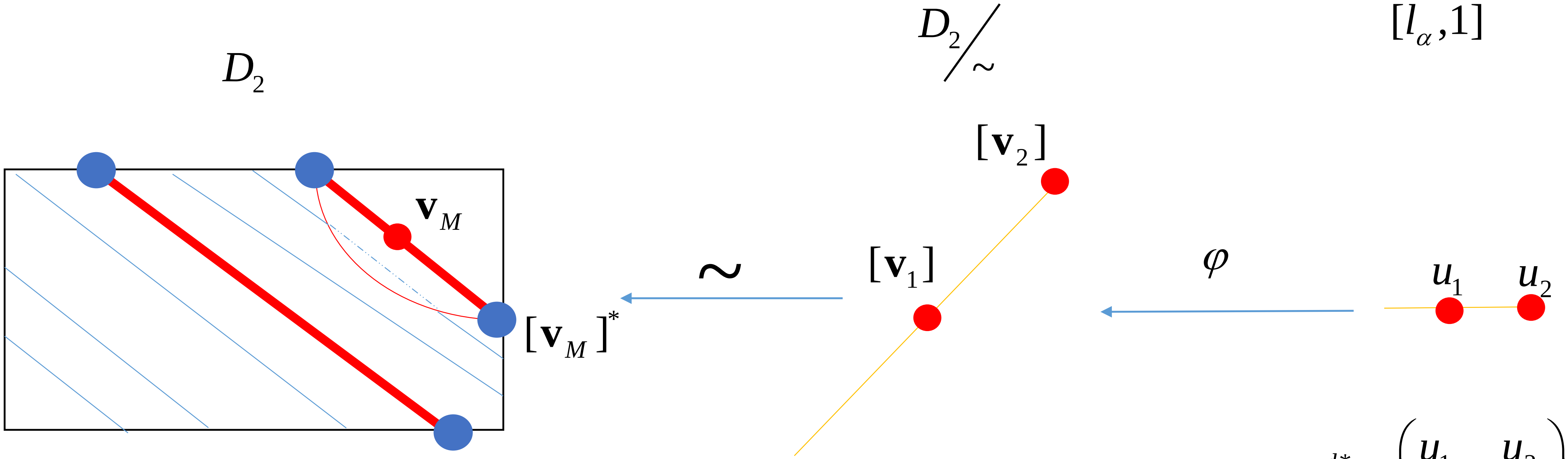
$$\zeta^{d*} = \begin{pmatrix} u_1 & u_2 \\ p_1 & p_2 \end{pmatrix}$$

$$\zeta^{o} = \begin{pmatrix} \mathbf{v}_{11} & \mathbf{v}_{12} & \mathbf{v}_{21} \\ p_{11} & p_{12} & p_{21} \end{pmatrix}$$

$$\sum_{r=1}^{r_l} p_{lr} x_{lr} = p_l u_l x_M \ \ and \ \ \sum_{r=1}^{r_l} p_{lr} y_{lr} = p_l u_l y_M$$

$$G^o(\zeta^o) = G^d(\zeta^d)$$

Theorem 2

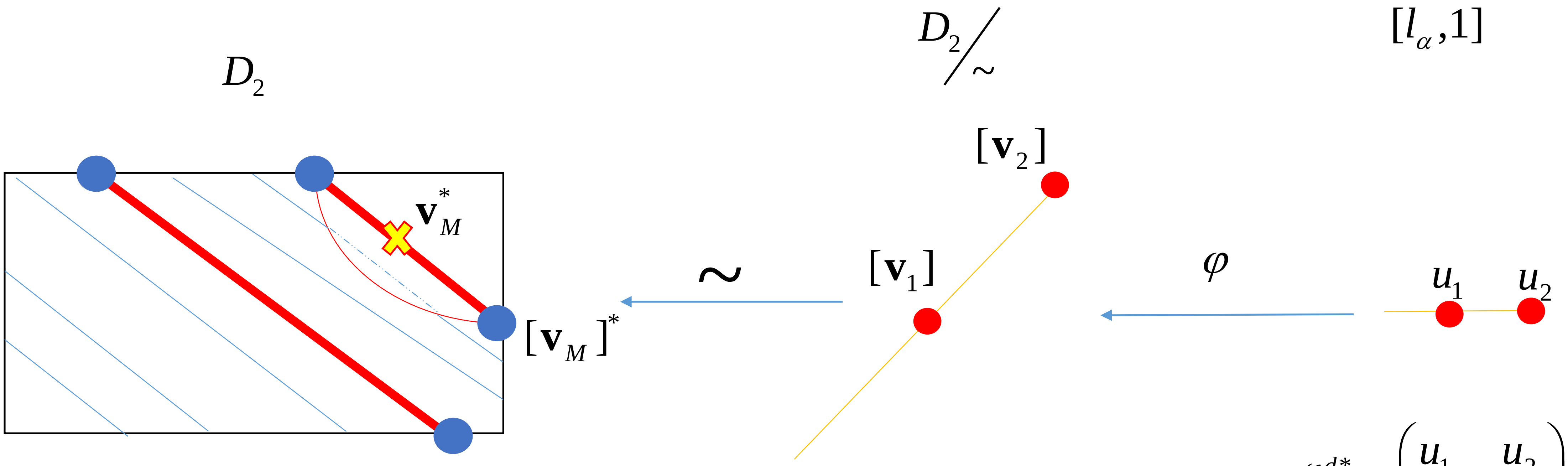


$$\zeta^{d*} = \begin{pmatrix} u_1 & u_2 \\ p_1 & p_2 \end{pmatrix}$$

$$\zeta^o = \begin{pmatrix} \mathbf{v}_{11} & \mathbf{v}_{12} & \mathbf{v}_{21} & \mathbf{v}_{22} \\ p_{11} & p_{12} & p_{21} & p_{22} \end{pmatrix}$$

For fixed $\mathbf{v}_M$, ζ^o has the largest determinant of the Fisher information matrix.

Theorem 3



$$\zeta^{d*} = \begin{pmatrix} u_1 & u_2 \\ p_1 & p_2 \end{pmatrix}$$

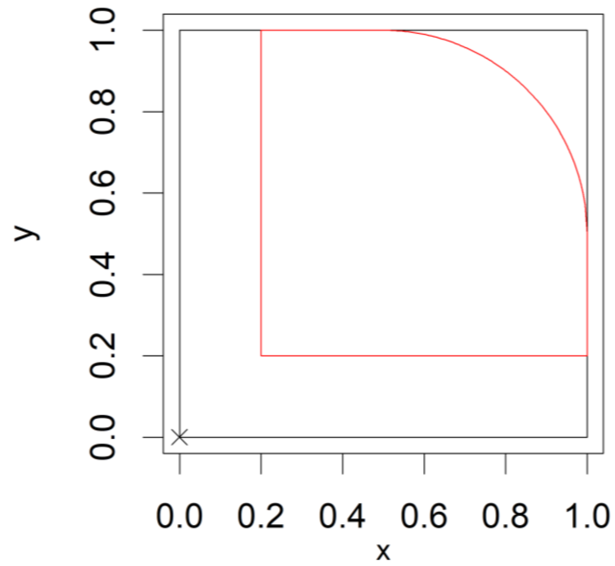
$$\zeta^o = \begin{pmatrix} \mathbf{v}_{11} & \mathbf{v}_{12} & \mathbf{v}_{21} & \mathbf{v}_{22} \\ p_{11} & p_{12} & p_{21} & p_{22} \end{pmatrix}$$

ζ^o has the largest determinant of the Fisher information matrix among all V-optimal design.

An illustrative example

- $d = 1.4$ and $(\alpha_0, \lambda, \alpha_1, \alpha_2)=(0.5,0.05,6,4)$
- $\mathcal{D}_2^{(3)} = \{\mathbf{v} \in [0.2,1]^2 | (x-0.5)^2 + (y-0.5)^2 \leq 0.25 \text{ if } x \in [0.5,1] \text{ and } y \in [0.5,1]\}$
- $\mathbf{v}_m = (0.2, 0.2)$
- $\mathbf{v}_M = (0.92, 0.78)$ can be obtained by the method of Lagrange multiplier

$$6x + 4y - \lambda((x-0.5)^2 + (y-0.5)^2 - 0.25)$$



(c)

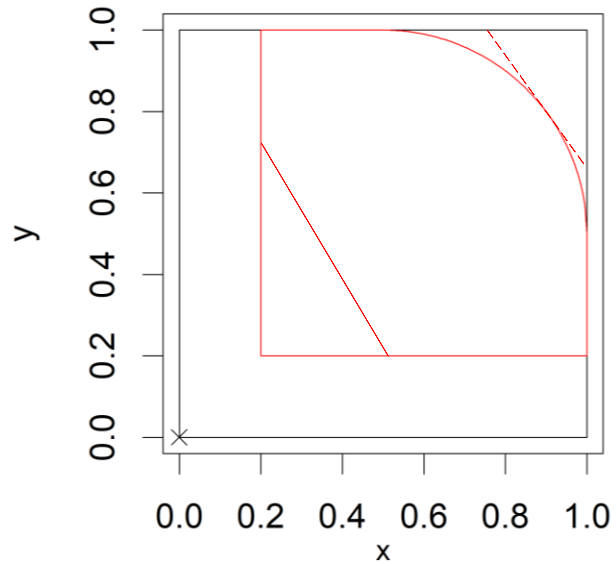
An illustrative example

- $d = 1.4$ and $(\alpha_0, \lambda, \alpha_1, \alpha_2) = (0.5, 0.05, 6, 4)$
- $\mathcal{D}_2^{(3)} = \{\mathbf{v} \in [0.2, 1]^2 | (x - 0.5)^2 + (y - 0.5)^2 \leq 0.25 \text{ if } x \in [0.5, 1] \text{ and } y \in [0.5, 1]\}$
- $\mathbf{v}_m = (0.2, 0.2)$
- $\mathbf{v}_M = (0.92, 0.78)$ can be obtained by the method of Lagrange multiplier

$$6x + 4y - \lambda((x - 0.5)^2 + (y - 0.5)^2 - 0.25)$$

- $l_\alpha = 0.23$ and $\alpha_1' = 8.6$

$$\zeta^{d*} = \begin{pmatrix} 0.51 & 1 \\ 0.88 & 0.12 \end{pmatrix}$$



(c)

An illustrative example

- $d = 1.4$ and $(\alpha_0, \lambda, \alpha_1, \alpha_2) = (0.5, 0.05, 6, 4)$
- $\mathcal{D}_2^{(3)} = \{\mathbf{v} \in [0.2, 1]^2 | (x - 0.5)^2 + (y - 0.5)^2 \leq 0.25 \text{ if } x \in [0.5, 1] \text{ and } y \in [0.5, 1]\}$
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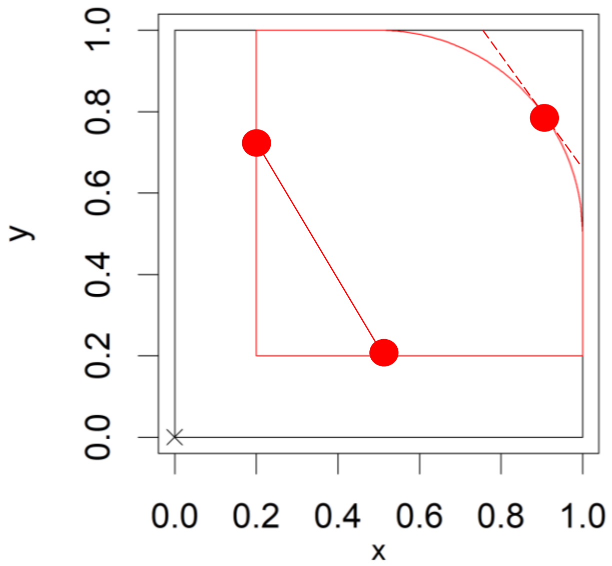
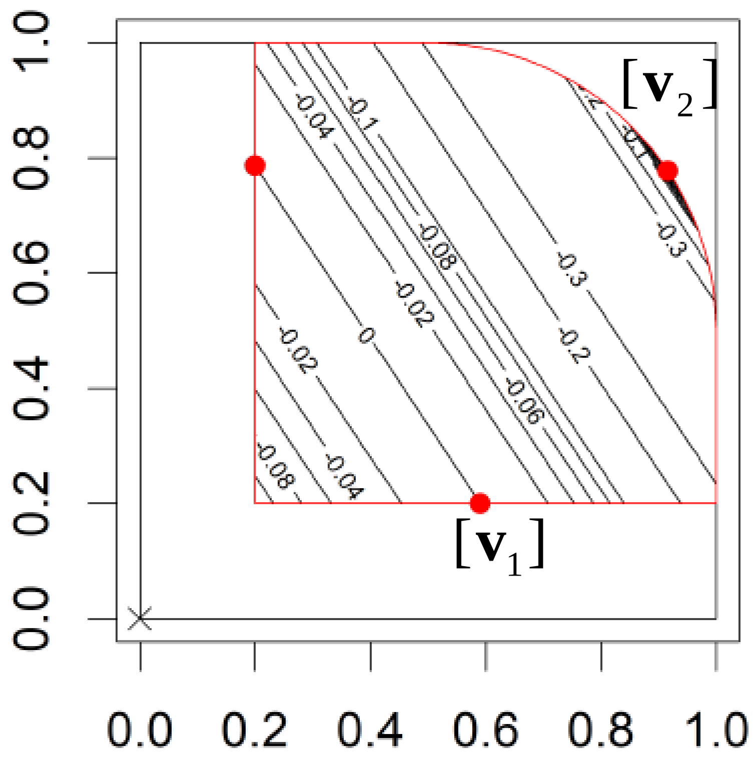


(c)

An illustrative example

From Theorem 1, 2 and 3

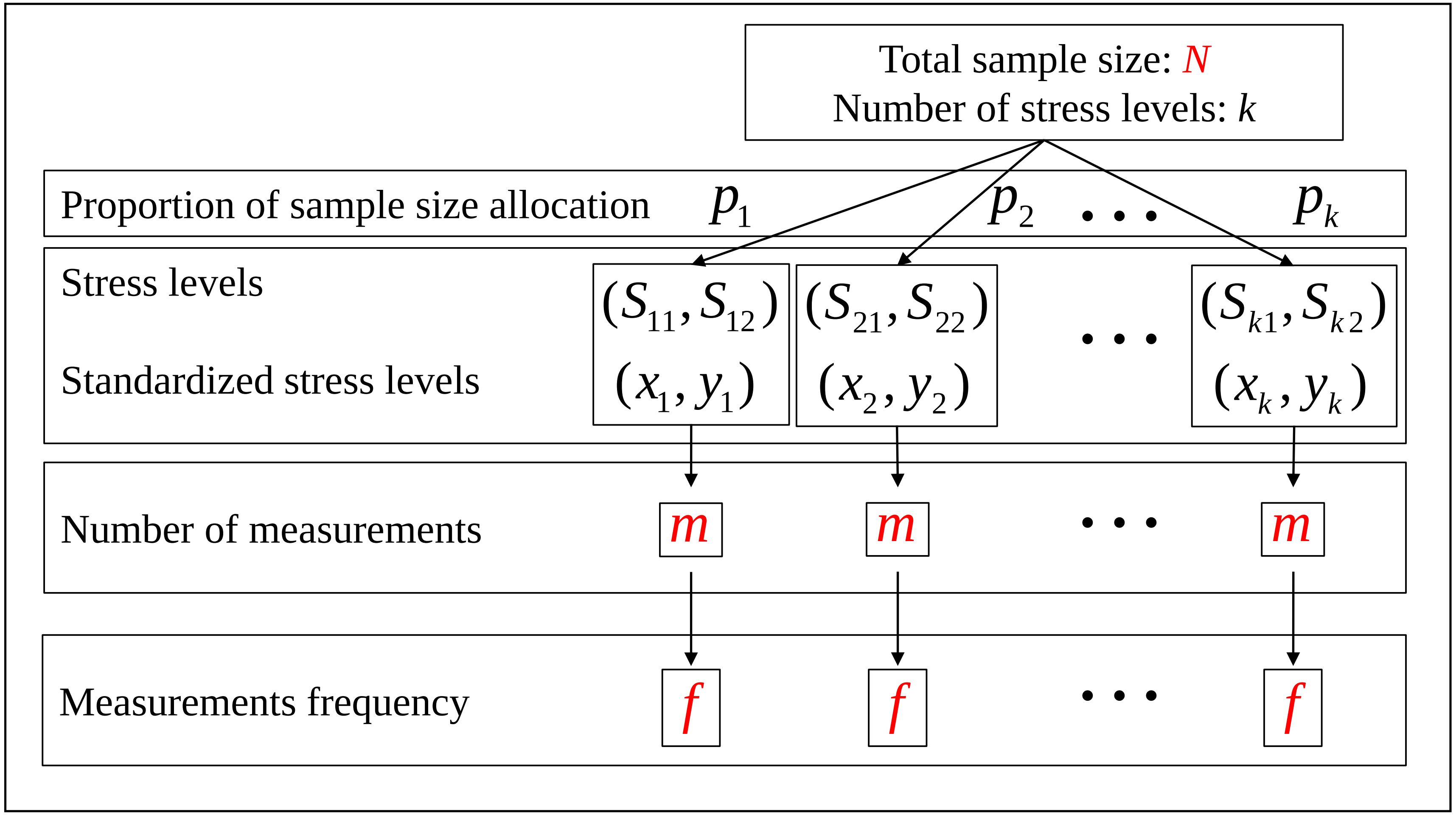
$$\begin{pmatrix} (0.2, 0.79) & (0.59, 0.2) & (0.92, 0.78) \\ 0.29 & 0.59 & 0.12 \end{pmatrix}$$



- Optimal design for EDADTs of two accelerating variables with interaction

  - Problem formulation
  - The expression of Avar $(\hat{\xi}_q)$
  - The Conjecture design under $d = 2$
  - The Conjecture design under $d < 2$
  - The Conjecture design under $d > 2$
  - LED example and numerical validation

Problem formulation



$$\zeta = \begin{pmatrix} (x_1, y_1) & \cdots & (x_k, y_k) \\ p_1 & \cdots & p_k \end{pmatrix}$$

Problem formulation

- Let $Z_i(t_j|x_l)$ $(i=1,\ldots,n_i,\ j=1,\ldots m,\ l=1,\ldots,k)$ denote the degradation of ith test unit at time $t_j = j\times\Delta t$ under lth stress-level (x_l, y_l).

- $Z_i(t_j|x_l) \sim ED(\mu(x_l, y_l)t_j, \lambda)$, $\ln(\mu(x_l, y_l)) = \alpha_0 + \alpha_1 x_l + \alpha_2 y_l + \alpha_3 x_l y_l$, $\alpha_1, \alpha_2, \alpha_3 > 0$, $(x_l, y_l) \in [l_x, 1]\times[l_y, 1]$

  $\Delta Z_{ijl} = Z_i(t_j|x_l) - Z_i(t_{(j-1)}|x_l)$ has the probability density function:

$$f(\Delta z_{ijl} \mid \mu(x_l, y_l), \lambda) = c(\Delta z_{ijl} \mid \lambda, \Delta t)e^{\lambda\{\varpi(\mu(x_l,y_l))\Delta z_{ijl} - \Delta t\kappa[\varpi(\mu(x_l,y_l))]\}}$$

The expression of Avar $(\hat{\xi}_q)$

- Asymptotic variance

$$\mathrm{AVar}(\hat{\xi}_q \mid \zeta) = \frac{1}{f_T(\xi_q;\boldsymbol{\theta})^2}\left(\boldsymbol{v}^{\mathrm{T}}\mathcal{I}^*(\boldsymbol{\theta},\zeta)^{-1}\boldsymbol{v}\right)$$

- To minimize AVar$(\hat{\xi}_q|\zeta)$ is equivalent to minimize $\psi(\zeta)$.

$$\Psi(\zeta) = \boldsymbol{c}^{\mathrm{T}}\mathcal{I}(\zeta)^{-1}\boldsymbol{c}. \qquad \mathcal{I}(\zeta) = \sum_{l=1}^{k} A_l p_l \begin{pmatrix} 1 \\ x_l \\ y_l \\ x_l y_l \end{pmatrix}\begin{pmatrix} 1 \\ x_l \\ y_l \\ x_l y_l \end{pmatrix}^T \qquad A_l = e^{-(d-2)(\alpha_0+\alpha_1 x_l+\alpha_2 y_l+\alpha_3 x_l y_l)}$$

$$\boldsymbol{c} = (1,0,0,0)^{\mathrm{T}}$$

The expression of Avar $(\hat{\xi}_q)$

Theorem

$$\Psi(\zeta \mid k=4) = \frac{1}{(-s_1+s_2-s_3+s_4)^2}\left(\frac{A_1^{-1}s_1^2}{p_1}+\frac{A_2^{-1}s_2^2}{p_2}+\frac{A_3^{-1}s_3^2}{p_3}+\frac{A_4^{-1}s_4^2}{p_4}\right),$$

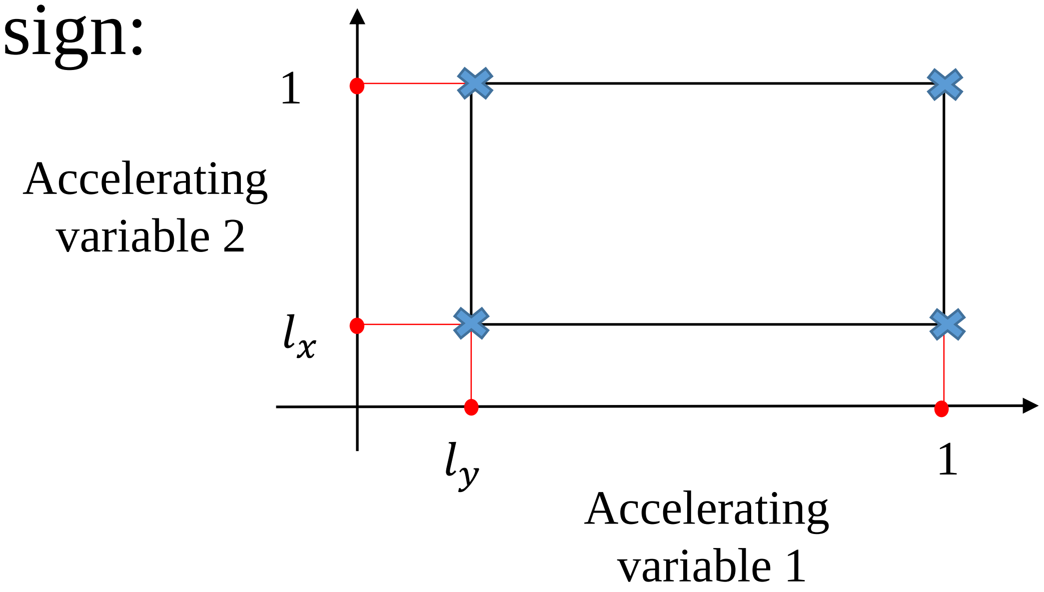
where

$$s_l = det\begin{bmatrix} x_{l_1} & x_{l_2} & x_{l_3} \\ y_{l_1} & y_{l_2} & y_{l_3} \\ x_{l_1}y_{l_1} & x_{l_2}y_{l_2} & x_{l_3}y_{l_3} \end{bmatrix}, \quad l_1 < l_2 < l_3 \in \{1,2,3,4\} \setminus \{l\}.$$

$$p_l^{\Delta}(x,y) = \frac{\sqrt{A_l^{-1}}|s_l|}{\sqrt{A_1^{-1}}|s_1|+\sqrt{A_2^{-1}}|s_2|+\sqrt{A_3^{-1}}|s_3|+\sqrt{A_4^{-1}}|s_4|} \quad (l=1,\cdots,4),$$

The Conjecture design under $d = 2$

- Step 1 : Guess optimal design:



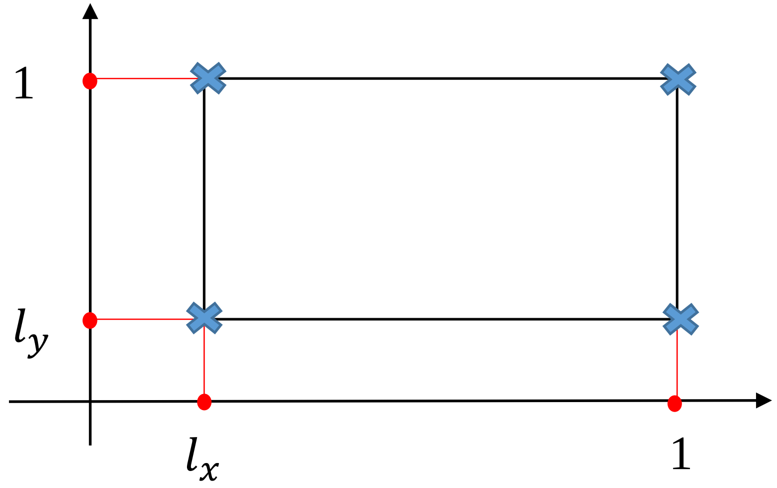
- Step 2 : Use General Equivalence Theorem to verify

The Conjecture design under $d = 2$

- The conjecture design is

$$\zeta^{\Delta} = \begin{pmatrix} (l_x, l_y) & (l_x, 1) & (1, l_y) & (1, 1) \\ \frac{1}{(1+l_x)(1+l_y)} & \frac{l_y}{(1+l_x)(1+l_y)} & \frac{l_x}{(1+l_x)(1+l_y)} & \frac{l_x l_y}{(1+l_x)(1+l_y)} \end{pmatrix}$$

The Conjecture design under $d = 2$



- Step 2: Set $\phi = -\Psi$

Lemma 1. *ϕ is a concave function.*

Lemma 2.

$$\Lambda(\zeta^{\Delta}, \zeta_{(x,y)}) = -\frac{4(1+l_x)^2(1+l_y)^2 c_1(x,y) c_2(x,y)}{(1-l_x)^4(1-l_y)^4},$$

where

$$c_1(x,y) = (x-l_x)(1-y) + (y-l_y)(1-x),$$

and

$$c_2(x,y) = (1-x)(1-y) + (x-l_x)(y-l_y).$$

The Conjecture design under $d = 2$

Theorem

$$\zeta^{\Delta} = \begin{pmatrix} (l_x, l_y) & (l_x, 1) & (1, l_y) & (1, 1) \\ \frac{1}{(1+l_x)(1+l_y)} & \frac{l_y}{(1+l_x)(1+l_y)} & \frac{l_x}{(1+l_x)(1+l_y)} & \frac{l_x l_y}{(1+l_x)(1+l_y)} \end{pmatrix}$$

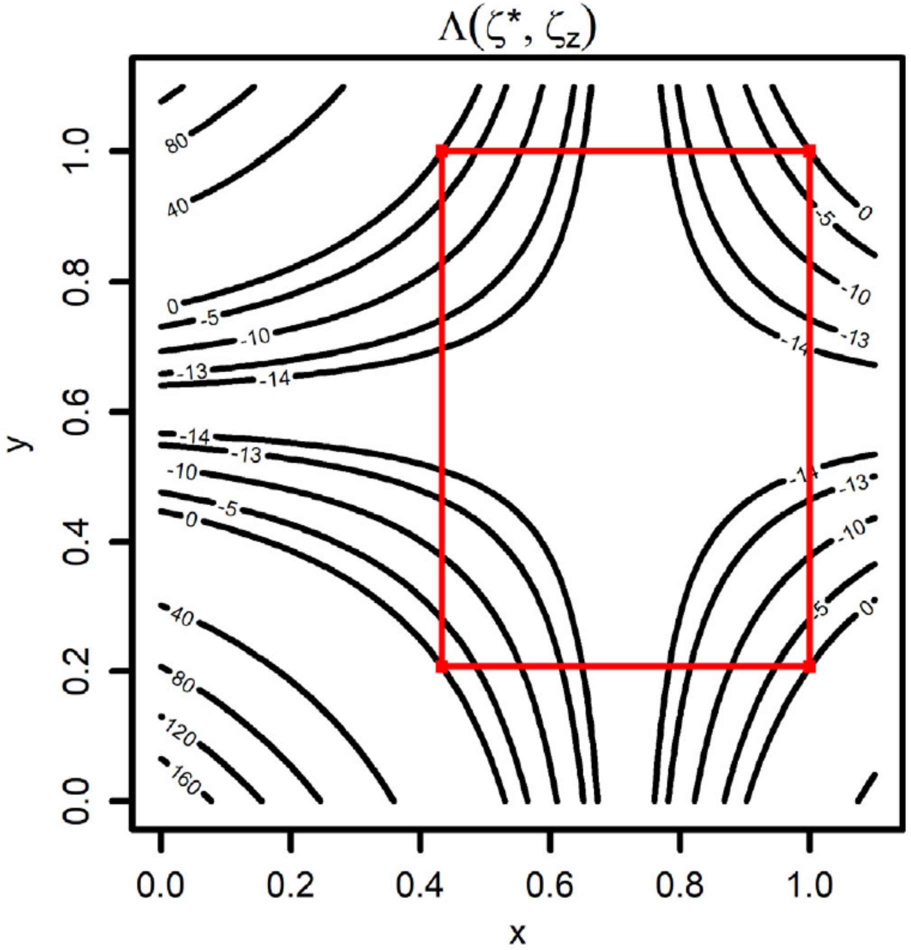
ζ^{Δ} is V-optimal design.

An illustrative example

- LED data (Tseng and Peng, 2007)
- Temperature: 45°C, 85°C
- Voltage: 10V, 30V
- Normal use condition: (20°C,7.5V)
- $l_x = 0.433, l_y = 0.208$

$$\zeta^* = \begin{pmatrix} (0.433, 0.208) & (1, 0.208) & (0.433, 1) & (1,1) \\ 0.578 & 0.250 & 0.120 & 0.052 \end{pmatrix}$$

An illustrative example

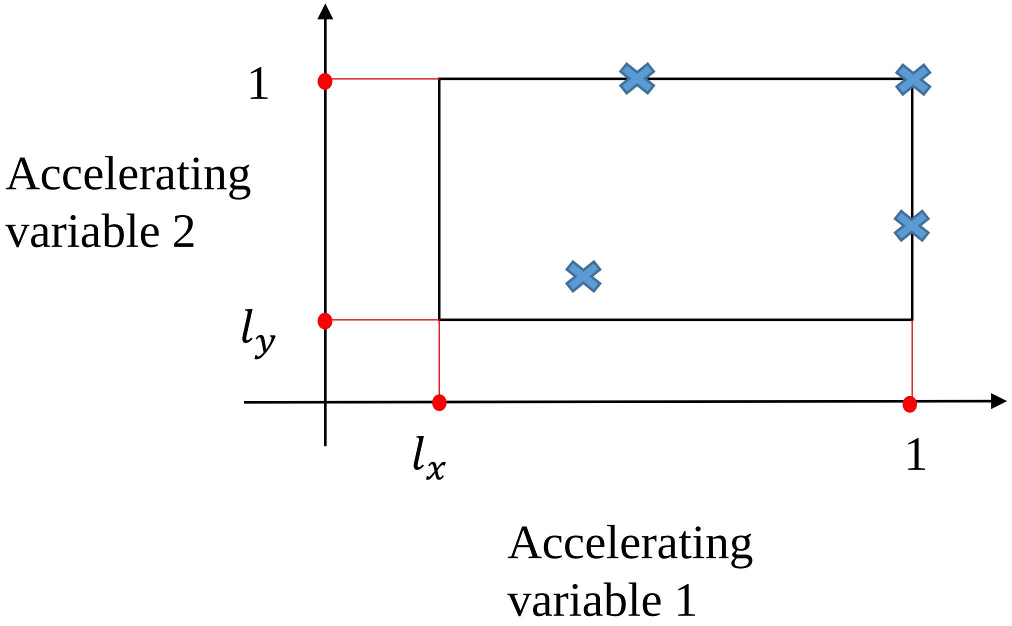


The Conjecture design under $d < 2$

$$\zeta^{\Delta} = \begin{pmatrix} (x_1^{\Delta}, y_1^{\Delta}) & (x_2^{\Delta}, 1) & (1, y_3^{\Delta}) & (1,1) \\ p_1^{\Delta}(x_1^{\Delta}, y_1^{\Delta}) & p_2^{\Delta}(x_2^{\Delta}, 1) & p_3^{\Delta}(1, y_3^{\Delta}) & p_4^{\Delta}(1,1) \end{pmatrix}$$

$$x_2^{\Delta} = \max(l_x, 1 + [1 + W(e^{-1})]\frac{2}{(d-2)(\alpha_1 + \alpha_3)}),$$

$$y_3^{\Delta} = \max(l_y, 1 + [1 + W(e^{-1})]\frac{2}{(d-2)(\alpha_2 + \alpha_3)}).$$

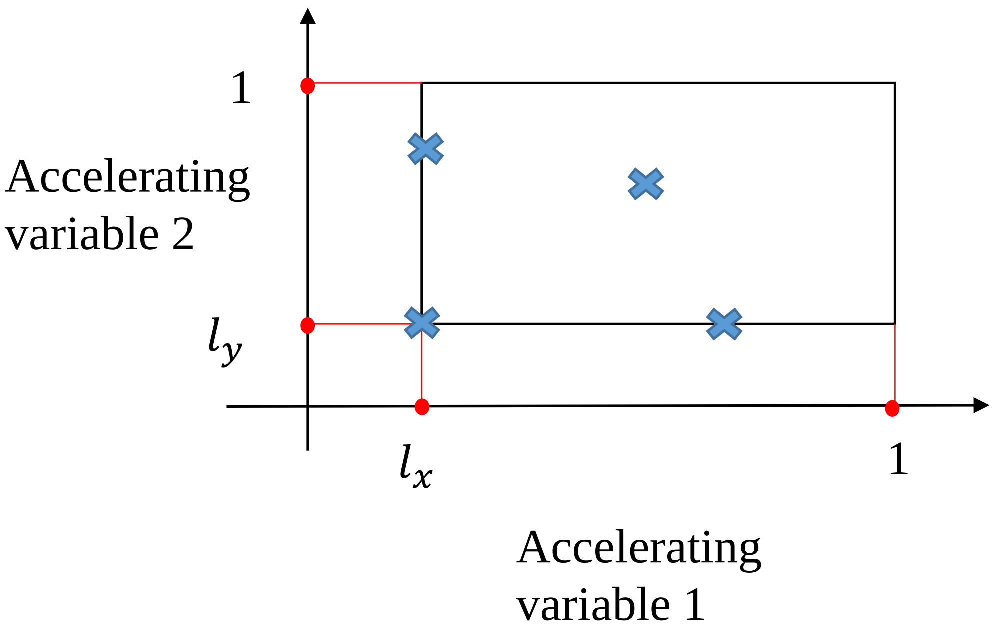


The Conjecture design under $d > 2$

$$\zeta^{\Delta} = \begin{pmatrix} (l_x, l_y) & (l_x, y_2^{\Delta}) & (x_3^{\Delta}, l_y) & (x_4^{\Delta}, y_4^{\Delta}) \\ p_1^{\Delta}(l_x, l_y) & p_2^{\Delta}(l_x, y_2^{\Delta}) & p_3^{\Delta}(x_3^{\Delta}, l_y) & p_4^{\Delta}(x_4^{\Delta}, y_4^{\Delta}) \end{pmatrix}$$

$$y_2^{\Delta} = \min(1, l_y + [1 + W(e^{-1})]\frac{2}{(d-2)(\alpha_2 + \alpha_3 l_x)});$$

$$x_3^{\Delta} = \max(1, l_x + [1 + W(e^{-1})]\frac{2}{(d-2)(\alpha_1 + \alpha_3 l_y)})$$



Numerical validation

Case 1: $d = 0$, $\alpha_1 = 1, 2, 3$, $\alpha_2 = 1, 2, 3$ and $\alpha_3 = 2$;

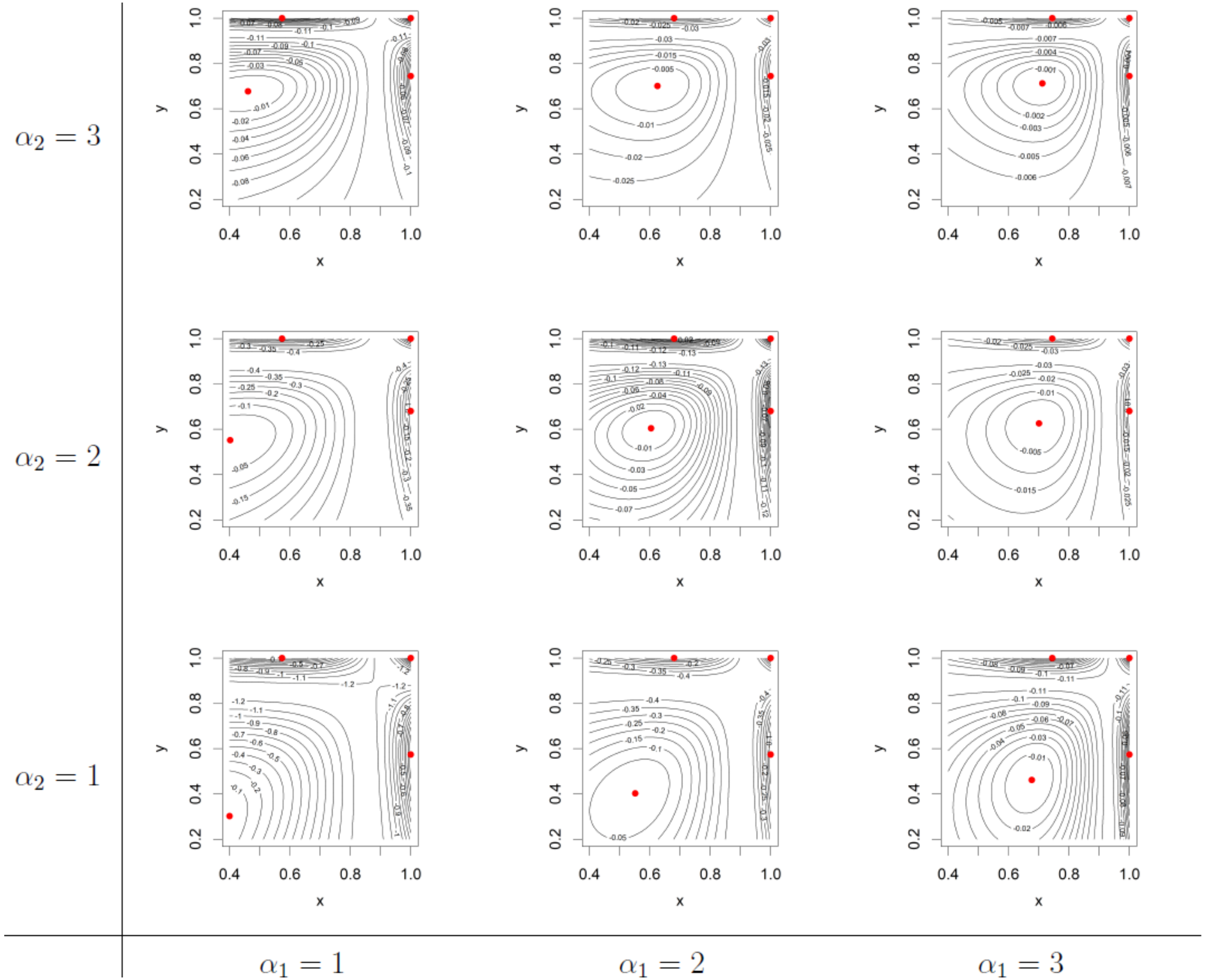
Case 2: $d = 0$, $\alpha_1 = 1$, $\alpha_2 = 1$ and $\alpha_3 = 1, 2, 3$;

Case 3: $d = 3.2$, $\alpha_1 = 2, 3, 4$, $\alpha_2 = 2, 3, 4$ and $\alpha_3 = 2$;

Case 4: $d = 3.2$, $\alpha_1 = 4$, $\alpha_2 = 3$ and $\alpha_3 = 1, 2, 3$;

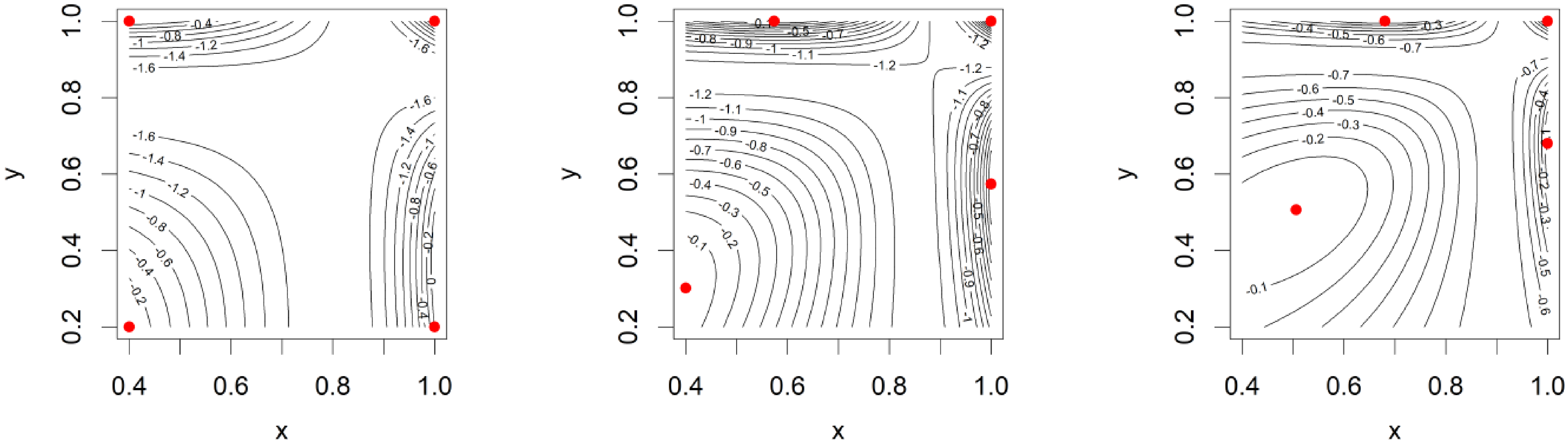
Numerical validation

Case 1:



Numerical validation

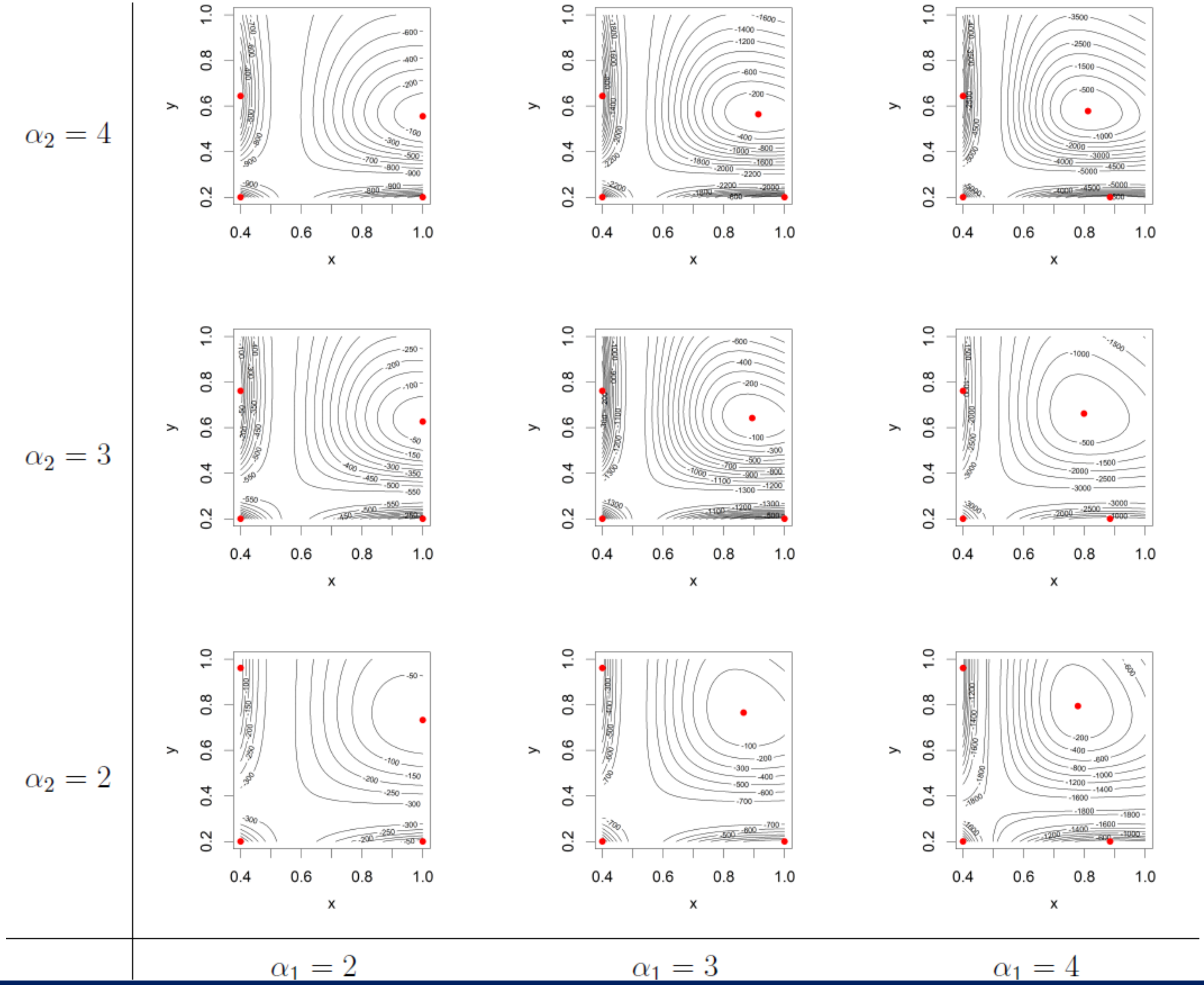
Case 2:



$\alpha_3 = 1$ $\alpha_3 = 2$ $\alpha_3 = 3$

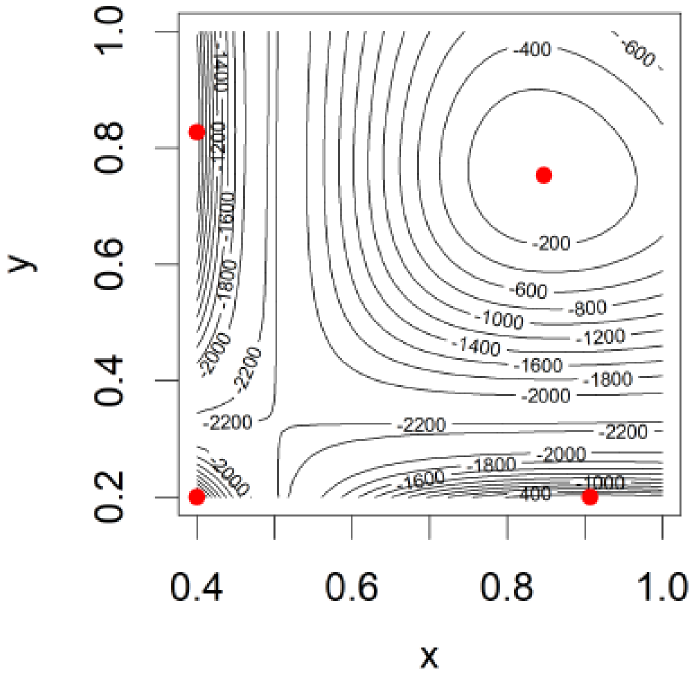
Numerical validation

Case 3:

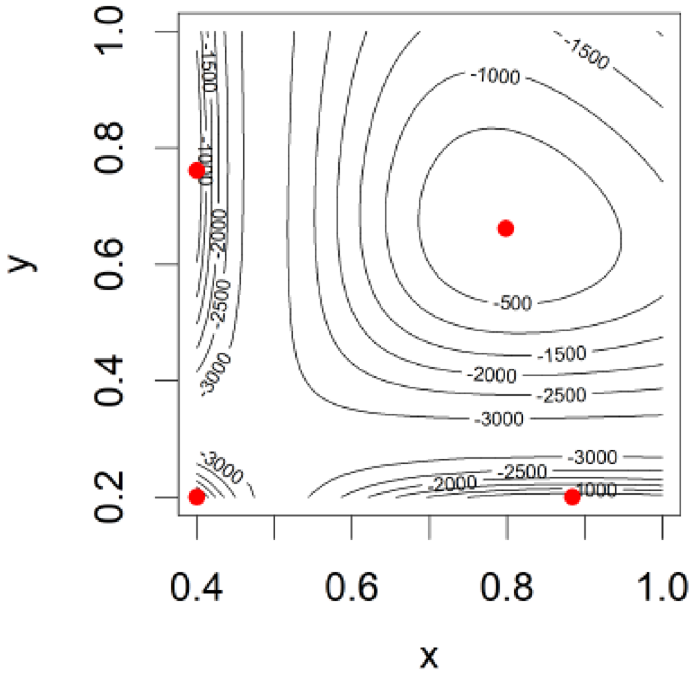


Numerical validation

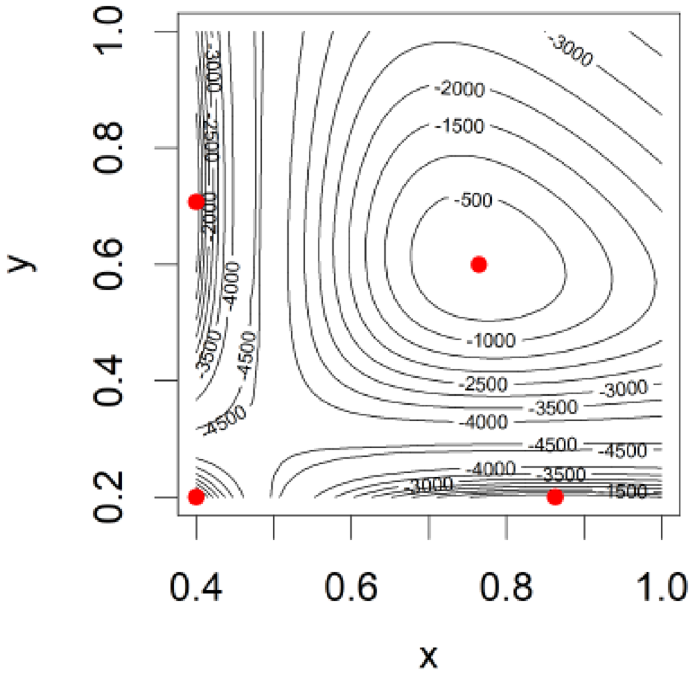
Case 4:



$\alpha_3 = 1$



$\alpha_3 = 2$



$\alpha_3 = 3$

Numerical validation

- Hong and Ye (2017) mentioned the necessity of acceleration.

- Coefficient of variation of EDADT

$$\frac{\mu(x,y)t}{\sqrt{\mu(x,y)^d t/\lambda}} = \mu(x,y)^{1-d/2} t^{1/2} \lambda^{-1/2}$$

Conclusion

- We provide a comprehensive study to the V-optimal ADT design problem when the underlying model follows an ED degradation model.
- We analytically prove the optimal design for EDADT with single accelerating variable.
- We analytically prove the optimal design for EDADT of two accelerating variables without interaction. Furthermore, the design region can be irregular.
- We analytically prove the optimal design for EDADT of two accelerating variables with interaction when $d = 2$. For $d \neq 2$, we proposed the conjecture designs and verify that the conjecture designs turn out to be the V -optimal design by the GET numerically.

Thank you for listening