Stable Matching: Why Interesting, Important and Fun?

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Objective of This Talk

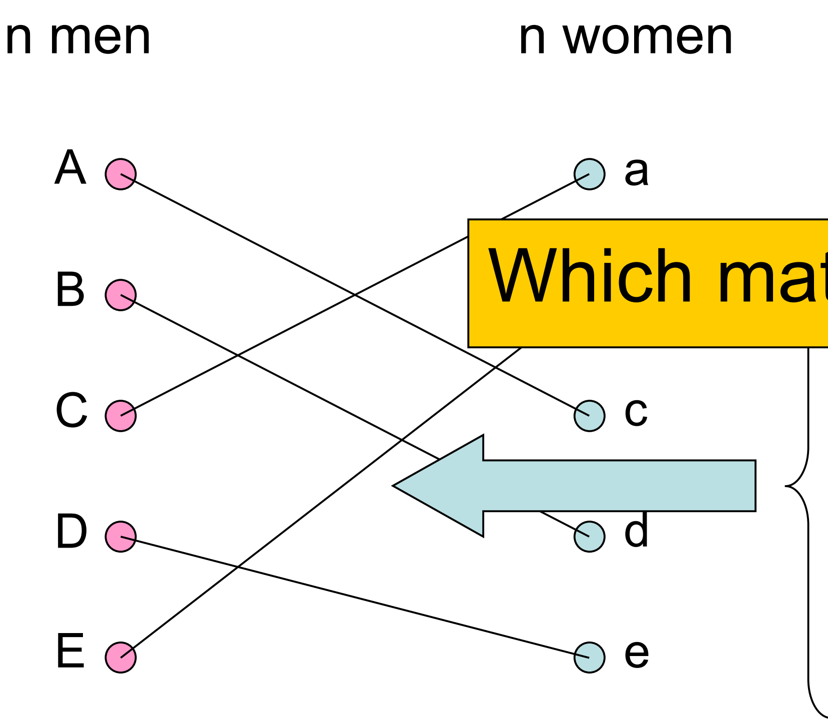
- What is the stable matching problem?
- Why is it interesting, important and fun?
- Using several (relatively old) results
- Some new results by our group

Stable Matching: Short History

- 1952: National Resident Matching Program (Assigning medical students to hospitals)
- 1962: D. Gale and L. Shapley. "College admissions and the stability of marriage"
- 1976: Donald E. Knuth, *Mariages Stables*, Les Presses de L'Universite de Montreal. FUN!
- 1989: D. Gusfield and R. W. Irving. *The Stable marriage Problem: Structure and Algorithms*, MIT
- 1990: A. Roth and M. Sotomayor. *Two-Sided Matching: A Study in Game Theoretic Modeling and An* IMPORTANT ridge
- 2012: A. Roth and L. Shapley. Nobel Prize in Economics
- 2013: D. Manlove. Algorithmics of Matching under Preferences

Stable Matching/Stable Marriage
Original Problem

Bipartite Matching



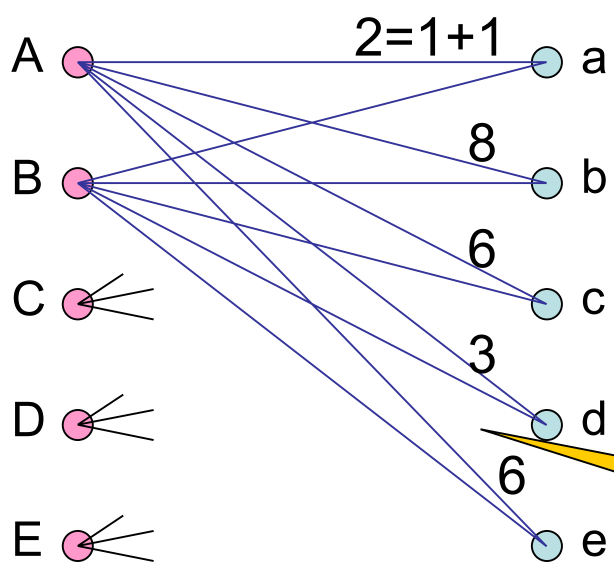
	a	b	c	d	e
A	1	5	3	2	4
B	1	4	3	5	2
C	3	4	2	5	1
D	[illegible]	[illegible]	[illegible]	[illegible]	5
E	[illegible]	[illegible]	[illegible]	[illegible]	4

Which matching is "good?"

	A	B	C	D	E
a	1	4	2	3	5
b	3	2	1	5	4
c	3	4	5	2	1
d	1	5	4	3	2
e	2	3	1	5	4

VERY IMPORTANT

Bipartite Matching



	a	b	c	d	e
A	1	5	3	2	4
B	1	4	3	5	2
C	3	4	2	5	1
D	1	2	4	3	5
E	2	1	3	5	4

	A	B	C	D	E
a	1	4	2	3	5
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
d	1	5	4	3	2
e	2	3	1	5	4

Minimum weight matching

Hungarian Method

MWM: Globally OK, but…

cost=10

1:	a	c	b	a:	1	3	2
2:	b	a	c	b:	2	1	3
3:	a	b	c	c:	1	2	3

cost=8

1:	a	c	b	a:	1	3	2
2:	b	a	c	b:	2	1	3
3:	a	b	c	c:	1	2	3

Stability of Matching

[Gale, Shapley 62]

Blocking pair

1:	a	c	b	d	e
2:	c	a	e	b	d
3:	b	a	e	d	c
4:	c	b	d	e	a
5:	c	d	b	e	a

a:	2	1	3	4	5
b:	2	1	4	5	3
c:	1	2	3	5	4
d:	3	1	4	2	5
e:	4	3	1	2	5

Stable Matching

[Gale, Shapley 62]

1:	a	c	b	d	e	a:	2	1	3	4	5
2:	c	a	e	b	d	b:	2	1	4	5	3
3:	b	a	e	d	c	c:	1	2	3	5	4
4:	c	b	d	e	a	d:	3	1	4	2	5
5:	c	d	b	e	a	e:	4	3	1	2	5

To Remove Blocking Pairs…

1:	a	c	b	d	e	a:	2	1	3	4	5
2:	c	a	e	b	d	b:	2	1	4	5	3
3:	b	a	e	d	c	c:	1	2	3	5	4
4:	c	b	d	e	a	d:	3	1	4	2	5
5:	c	d	b	e	a	e:	4	3	1	2	5

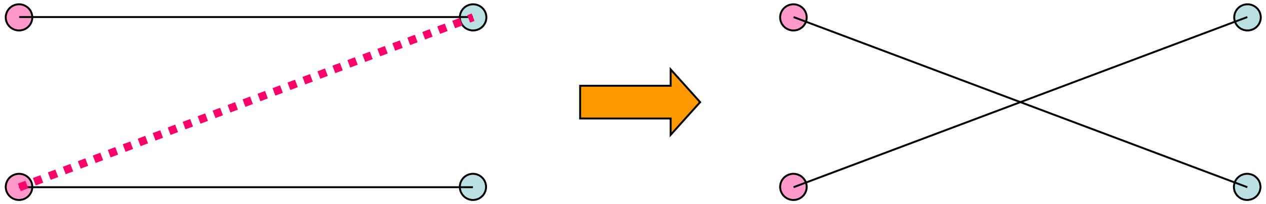
No blocking pairs any more

Possible Algorithm for Obtaining a Stable Matching

Let M = any matching

While M includes blocking pairs

Do select any such a pair and swap



End

Output M

Conjectured in [Knuth 76]

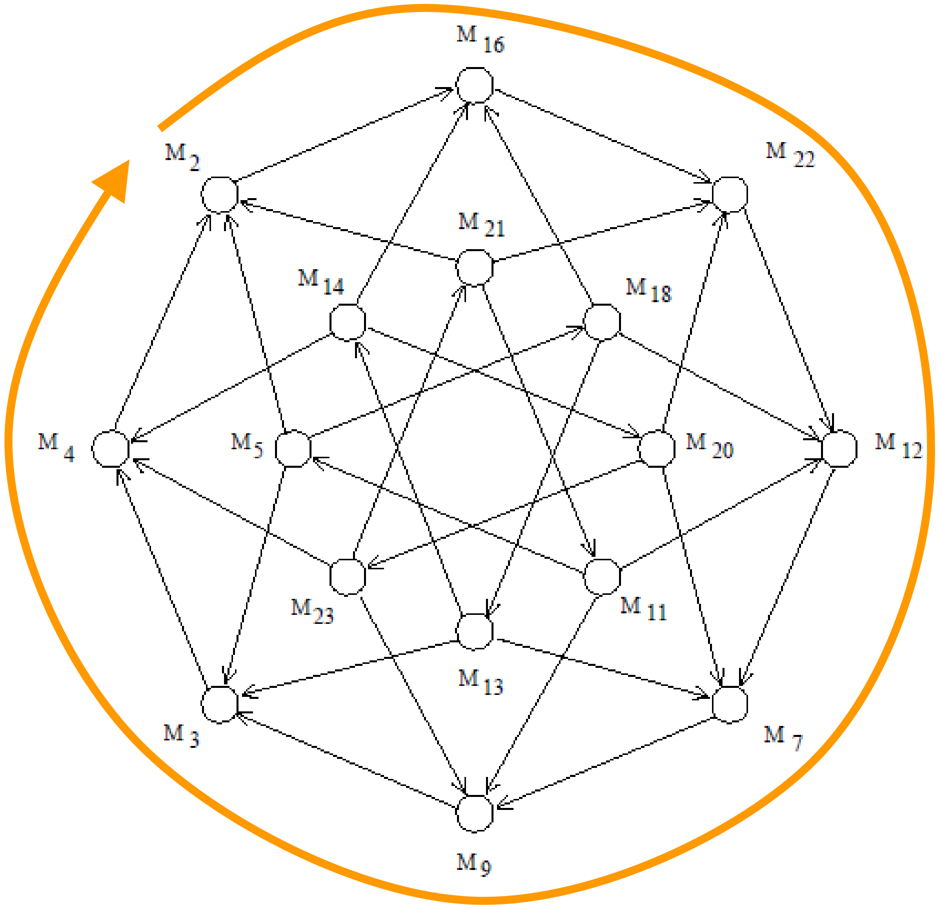
However,…

[Tamura 93]

1:	a	c	b	d	a:	2	4	1	3
2:	b	d	c	a	b:	3	1	2	4
3:	c	a	d	b	c:	4	2	3	1
4:	d	b	a	c	d:	1	3	4	2

1:	a	c	b	d	a:	2	4	1	3
2:	b	d	c	a	b:	3	1	2	4
3:	c	a	d	b	c:	4	2	3	1
4:	d	b	a	c	d:	1	3	4	2

It Loops



FUN!!

GS Algorithm

[Gale, Shapley 62]

1:	a	c	b	d	e	a:	2	1	3	4	5
2:	c	a	e	b	d	b:	2	1	4	5	3
3:	b	a	e	d	c	c:	1	2	3	5	4
4:	c	b	d	e	a	d:	3	1	4	2	5
5:	c	d	b	e	a	e:	4	3	1	2	5

Amazing Theorem: GS always finds a stable matching.

∵ no partner => 4: X X X X X => e: 3 1 4 2 5 (no ◯)

no blocking pair similarly.

An Important Operation for SMs

Is a stable matching unique? No.

1:	a	c	b	d	a:	2	4	1	3
2:	b	d	c	a	b:	3	1	2	4
3:	c	a	d	b	c:	4	2	3	1
4:	d	b	a	c	d:	1	3	4	2

1:	a	c	b	d	a:	2	4	1	3
2:	b	d	c	a	b:	3	1	2	4
3:	c	a	d	b	c:	4	2	3	1
4:	d	b	a	c	d:	1	3	4	2

1:	a	c	b	d	a:	2	4	1	3
2:	b	d	c	a	b:	3	1	2	4
3:	c	a	d	b	c:	4	2	3	1
4:	d	b	a	c	d:	1	3	4	2

1:	a	c	b	d	a:	2	4	1	3
2:	b	d	c	a	b:	3	1	2	4
3:	c	a	d	b	c:	4	2	3	1
4:	d	b	a	c	d:	1	3	4	2

An Important Operation for SMs

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1:	a	c	b	d	a:	2	4	1	3
2:	b	d	c	a	b:	3	1	2	4
3:	c	a	d	b	c:	4	2	3	1
4:	d	b	a	c	d:	1	3	4	2

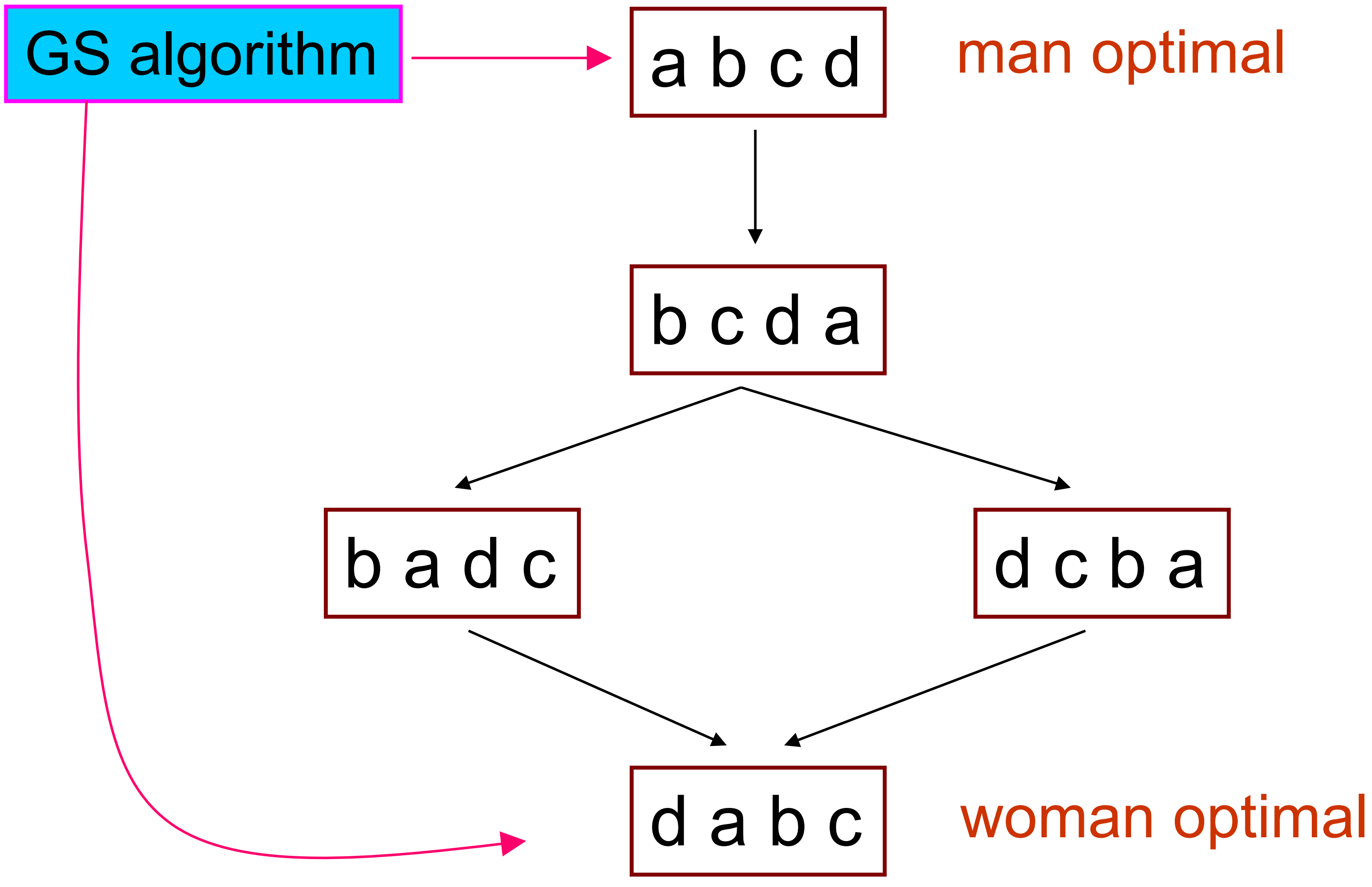
1:	a	c	b	d	a:	2	4	1	3
2:	b	d	c	a	b:	3	1	2	4
3:	c	a	d	b	c:	4	2	3	1
4:	d	b	a	c	d:	1	3	4	2

1:	a	c	b	d	a:	2	4	1	3
2:	b	d	c	a	b:	3	1	2	4
3:	c	a	d	b	c:	4	2	3	1
4:	d	b	a	c	d:	1	3	4	2

1:	a	c	b	d	a:	2	4	1	3
2:	b	d	c	a	b:	3	1	2	4
3:	c	a	d	b	c:	4	2	3	1
4:	d	b	a	c	d:	1	3	4	2

Rotation

Lattice Structure



Obtaining a "Good" Stable Matching

- GS algorithm => Man-opt or Woman-opt
- Egalitarian: minimizing sum of ranks (cost)
- Min-Regret: minimizing max of ranks
- Sex-Equal: minimizing diff of total ranks between men and women (NP-hard)

	1	2	3	4		1	2	3	4
1:	a	c	b	d	a:	2	4	1	3
2:	b	d	c	a	b:	3	1	2	4
3:	c	a	d	b	c:	4	2	3	1
4:	d	b	a	c	d:	1	3	4	2

⇐ rank

GS Works for Several Extensions

1:	**a**	c	b	f	d	e
2:	**c**	a	f	e	b	d
3:	b	a	**e**	d	c	f
4:	c	**b**	d	e	a	f
5:	c	**f**	d	b	e	a

a:	2	**1**	3	4	5
b:	2	1	**4**	5	3
c:	1	**2**	3	5	4
d:	3	1	4	2	5
e:	4	**3**	1	2	5
f:	1	3	2	**5**	4

Men: If free, propose to the currently best woman (in any order)

Women: Accept the propose if not oversubscribed and reject the worst otherwis

GS Works for Several Extensions

1:	a	c	b	f		
2:	c	a	f	e	b	d
3:	b	a				
4:	c	b	d	e	a	f
5:	c	f				

a:	2	1			
b:	2	1	4	5	3
c:	1	2	3		
d:	3	1	4	2	
e:	4	3	1	2	5
f:	1	3	2		

Men: Propose to the currently best woman (in any order)
Women: Accept the propose if not oversubscribed and reject the worst otherwis

GS Works for Several Extensions

1:	a	c	b	a:	2	1	3	4	5
2:	c	a	b	b:	2	1	4	5	3
3:	b	a	c	c:	1	2	3	5	4
4:	c	b	a						
5:	b	c	a						

Men: Propose to the currently best woman (in any order)
Women: Accept the propose if not oversubscribed and reject the worst otherwis

GS Works for Several Extensions

1:	a	c	b
2:	c	a	b
3:	b	a	c
4:	c	b	a
5:	b	c	a

a [3]:	2	1	3	4	5
b [2]:	2	1	4	5	3
c [1]:	1	2	3	5	4

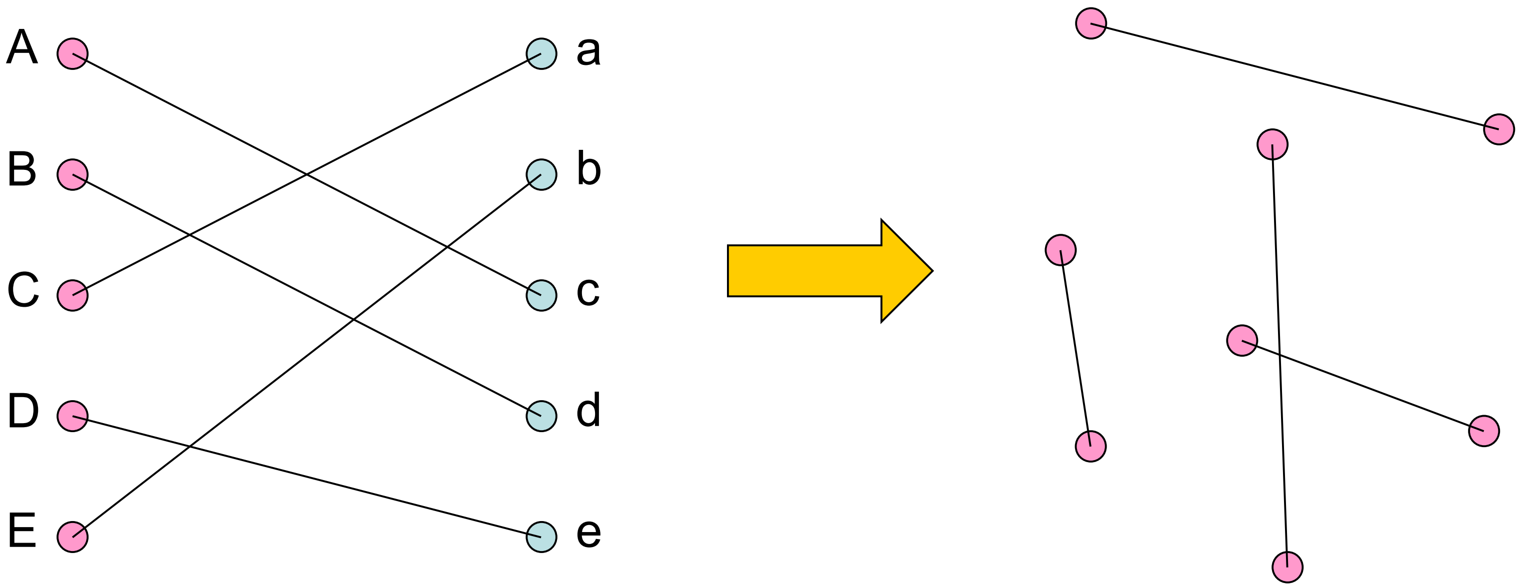
Residents hospital problem

Rural hospital theorem: Stable matchings may not be unique, but # of residents assigned to each hospital is unique.

Stable Roommate Problem

Stable Roommate Problem

[Gale, Shapley 62, and Knuth 76]



Stable roommates

1:	2	3	4
2:	3	4	1
3:	4	1	2
4:	2	3	1

1:	2	3	4
2:	3	4	1
3:	4	1	2
4:	2	3	1

1:	2	3	4
2:	3	1	4
3:	1	2	4
4:	1	2	3

Stable roommates

1:	**2**	3	4
2:	3	4	**1**
3:	**4**	1	2
4:	2	**3**	1

1:	~~2~~	**3**	4
2:	3	**4**	~~1~~
3:	4	**1**	2
4:	**2**	3	1

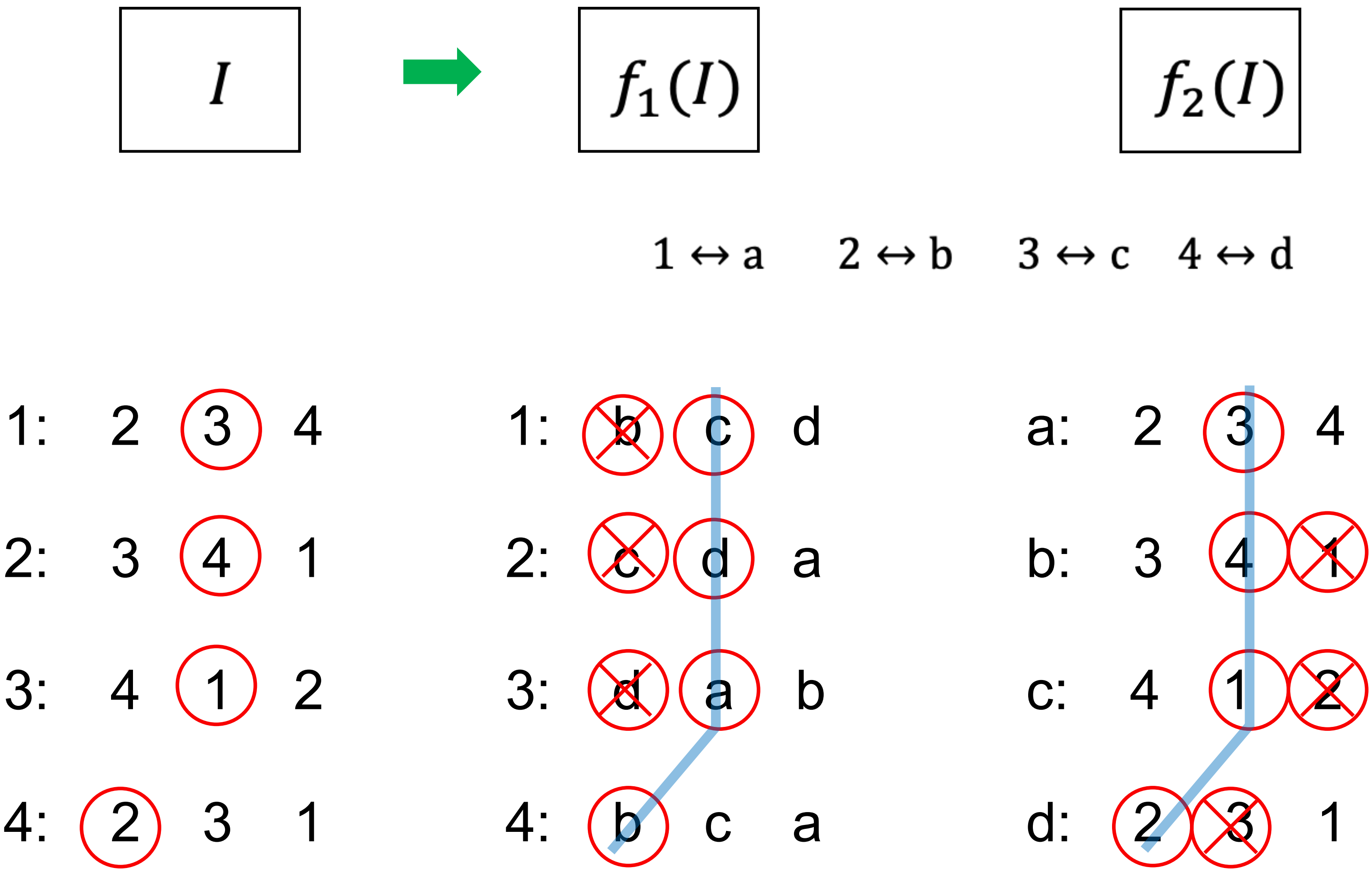
1:	2	**3**	4
2:	3	1	**4**
3:	**1**	2	4
4:	1	**2**	3

Stable roommates

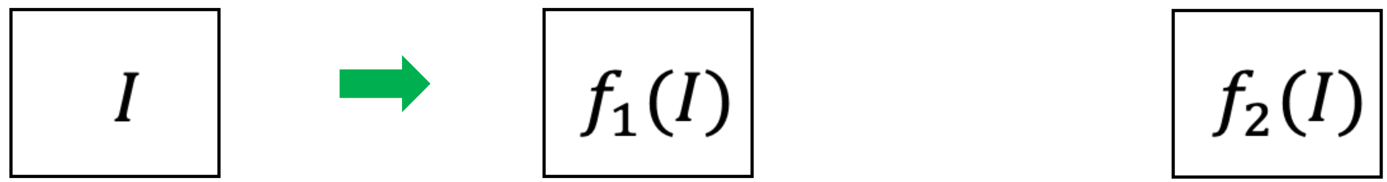
1:	2	3	4	1:	~~2~~	3	4	1:	2	3	4
2:	3	4	1	2:	3	4	~~1~~	2:	3	1	4
3:	4	1	2	3:	4	1	2	3:	1	2	4
4:	2	3	1	4:	2	3	1	4:	1	2	3

Still solvable in poly time [Irving 85]

Basic Ideas


I
$f_1(I)$
$f_2(I)$
$1 \leftrightarrow a$ $2 \leftrightarrow b$ $3 \leftrightarrow c$ $4 \leftrightarrow d$
1: 2 3 4
2: 3 4 1
3: 4 1 2
4: 2 3 1
1: b c d
2: c d a
3: d a b
4: b c a
a: 2 3 4
b: 3 4 1
c: 4 1 2
d: 2 3 1

Basic Ideas



$1 \leftrightarrow a \quad 2 \leftrightarrow b \quad 3 \leftrightarrow c \quad 4 \leftrightarrow d$

1:	2	3	4
2:	3	1	4
3:	4	1	2
4:	2	1	3

1:	b	c	d
2:	c	a	d
3:	d	a	b
4:	b	a	c

a:	2	3	4
b:	3	1	4
c:	4	1	2
d:	2	1	3

Basic Ideas

I → $f_1(I)$ $f_2(I)$

$1 \leftrightarrow a$ $2 \leftrightarrow b$ $3 \leftrightarrow c$ $4 \leftrightarrow d$

1:	2	3	4	1:	b	c	d	a:	2	3	~~4~~
2:	3	1	4	2:	c	a	d	b:	3	1	~~4~~
3:	1	2	4	3:	a	b	d	c:	1	2	~~4~~
4:	1	2	3	4:	~~a~~	~~b~~	~~c~~	d:	1	2	3

Basic Ideas

1:	2	5	3	6
2:	3	4	6	1
3:	4	1	5	2
4:	5	6	2	3
5:	6	3	1	4
6:	1	2	4	5

1:	b	e	c	f
2:	c	d	f	a
3:	d	a	e	b
4:	e	f	b	c
5:	f	c	a	d
6:	a	b	d	e

a:	2	5	3	6
b:	3	4	6	1
c:	4	1	5	2
d:	5	6	2	3
e:	6	3	1	4
f:	1	2	4	5

Approximation Algorithms
2000~

Relaxed Preference Lists

- Some matchmake site
  - Thousands of men and women!
- Complete total order is unrealistic

2: c a e b d

- Indifferences (ties) in the list

2: (c a) (e b d)

- Incomplete lists

2: c a e

Stable Matching with Incomplete List (SMI)

1:	a	c	b			a:	2	1	3	4	5
2:	c	a				b:	2	1			
3:	b	a				c:	1	2			
4:	c	b	d	e		d:	3	1	4		
5:	c	d	b			e:	4	3			

Matching may be partial

Stable Matching with Incomplete List (SMI)

1:	a	c	b			a:	2	1	3	4	5
2:	c	a				b:	2	1			
3:	b	a				c:	1	2			
4:	c	b	d	e		d:	3	1	4		
5:	c	d	b			e:	4	3			

Matching may be partial

Theorem [Gale, Sotomayor 1985] There may be more than one stable matchings, but their size is all the same and one of them can be obtained in poly time.

Theorem [Gusfield, Irving 1989, and Irving 1994] Any SMT instance admits at least one (weakly) stable matching, which can be obtained in poly time.

Theorem [Gale, Sotomayor 1985] An SMI instance may have more than one stable matching, but their size is all the same and one of them can be obtained in poly time.

Ties or Partial lists: Still OK
What if both are allowed?

SM with Ties and Incomplete Lists (SMTI)

1: a　　　　a: (1 2)

2: a b　　　b: 2

Stable matchings with different sizes

⇨ Problem of obtaining a max one (MAX SMTI)

Was open till 1999

MAX SMTI: Sequence of Results

- [I, Manlove, Miyazaki, Morita 99] MAX SMTI is NP-hard.
- Approximation Factor
- Approx ratio 2.0 is trivial because of maximal matching. But < 2.0 seems hard like min max matching and vertex cover

Is $2-\varepsilon$ possible?

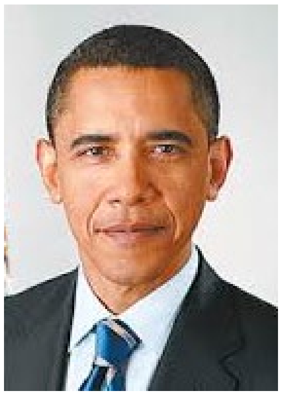
Approximation Upper Bounds

- [Halldorsson, I, Miyazaki, Yanagisawa 03]
  - 13/7 if the length of ties is two
- [Halldorsson, I, Miyazaki, Yanagisawa 04]
  - 7/4 (expected) if the length of tirs is two
- [I, Miyazaki, Okamoto 04] $2 - {c \log n}/{n}$
- [I, Miyazaki, Yamauchi 05] $2 - {c}/{\sqrt{n}}$
- [I, Miyazaki, Yamauchi 07] 15/8=1.85
- [Kiraly 08] 1.67
- [McDermid 09] 1.5

Game Theoretic Aspects

Stable Matching Is a Game

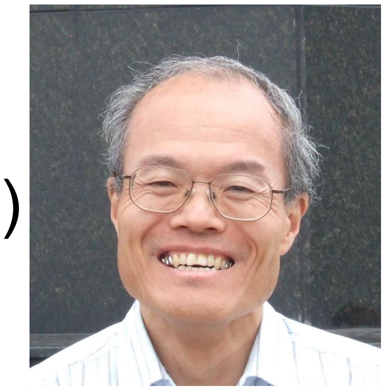
a>b>c>d

(1) 

b>a>c>d

(2) 

a>b>c>d

(3) 

c>d>a>b

(4) 

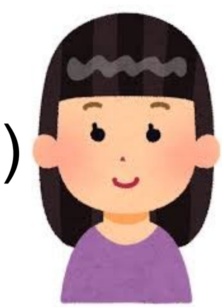
2>1>3>4

(a) 

1>2>3>4

(b) 

1>2>3>4

(c) 

3>4>1>2

(d) 

Strategy Proofness

1:	a	b	c	d	a:	2	1	3	4
2:	b	a	c	d	b:	1	2	3	4
3:	a	b	c	d	c:	1	2	3	4
4:	c	d	a	b	d:	4	3	1	2

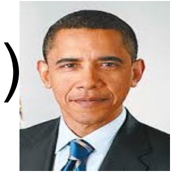
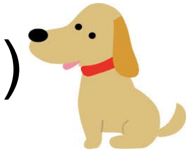
This list is not a Nash equilibrium

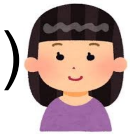
1:	a	b	c	d	a:	2	1	3	4
2:	b	a	c	d	b:	1	2	3	4
3:	b	a	c	d	c:	1	2	3	4
4:	c	d	a	b	d:	4	3	1	2

Men-propose GS is strategy proof for men.

1:	a	b	c	d	a:	2	3	1	4
2:	b	a	c	d	b:	1	2	3	4
3:	a	b	c	d	c:	1	2	3	4
4:	c	d	a	b	d:	4	3	1	2

Men-propose GS is not strategy proof for women.

(1) 
(2) 
(3) 
(4) 

(a) 
b) 
(c) 
(d) 

$P(M)$

1	a	b	d	c
2	a	b	c	d
3	a	c	b	d
4	a	b	c	d

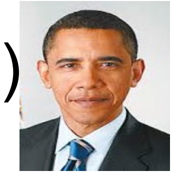
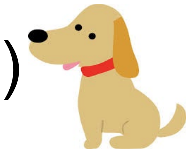
$P(W)$

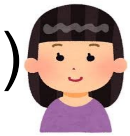
a	1	2	3	4
b	1	3	2	4
c	1	2	3	4
d	1	2	3	4

$Q(W)$

a	2	1	3	4
b	1	3	2	4
c	1	2	3	4
d	1	4	3	2

p-unstable

(1)  (2)  (3)  (4) 

(a)  b)  (c)  (d) 

$P(M)$

1	a	b	d	c
2	a	b	c	d
3	a	c	b	d
4	a	b	c	d

$P(W)$

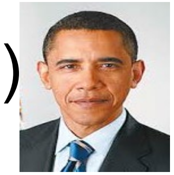
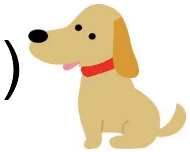
a	1	2	3	4
b	1	3	2	4
c	1	2	3	4
d	1	2	3	4

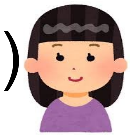
$Q(W)$

a	1	2	3	4
b	1	3	2	4
c	1	2	3	4
d	1	4	3	2

p-stable

Nash?

(1)  (2)  (3)  (4) 

(a)  b)  (c)  (d) 

$P(M)$

1	a	b	d	c
2	a	b	c	d
3	a	c	b	d
4	a	b	c	d

$P(W)$

a	1	2	3	4
b	1	3	2	4
c	1	2	3	4
d	1	2	3	4

$Q'(M)$

p-stable

a	1	2	3	4
b	1	3	2	4
c	2	1	4	3
d	1	4	3	2

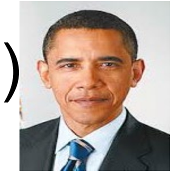
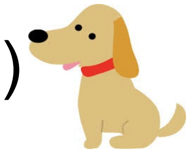
$Q(W)$

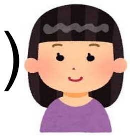
p-stable

Nash?

NO!

a	1	2	3	4
b	1	3	2	4
c	1	2	3	4
d	1	4	3	2

(1)  (2)  (3)  (4) 

(a)  b)  (c)  (d) 

$P(M)$

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a	1	2	3	4
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a	1	2	3	4
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$Q(W)$

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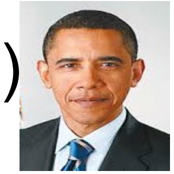
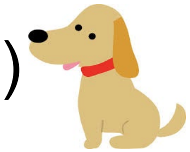
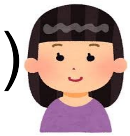
p-stable

Nash?

(1)  (2)  (3)  (4) 

(a)  b)  (c)  (d) 

[Gupta, I, Miyazaki 16]

$P(M)$

1	**a**	b	d	c
2	a	b	**c**	d
3	a	c	**b**	d
4	a	b	c	**d**

$P(W)$

a	**1**	2	3	4
b	1	2	**3**	4
c	1	**2**	3	4
d	1	2	3	**4**

$Q'(M)$

NOT p-stable

a	**1**	2	3	4
b	1	**3**	2	4
c	**2**	1	4	3
d	1	**4**	3	2

$Q(W)$

p-stable

Nash?

a	**1**	2	3	4
b	1	3	**2**	4
c	1	2	**3**	4
d	1	**4**	3	2

Effectively Nash

Stable Matching

- Started 1960s
- Nobel Prize!
- Constantly rising new interesting topics
  - Real applications
  - Residents/Hospital problems with lower quota
  - Popular matchings,
  - New angles from economics groups

Thank you

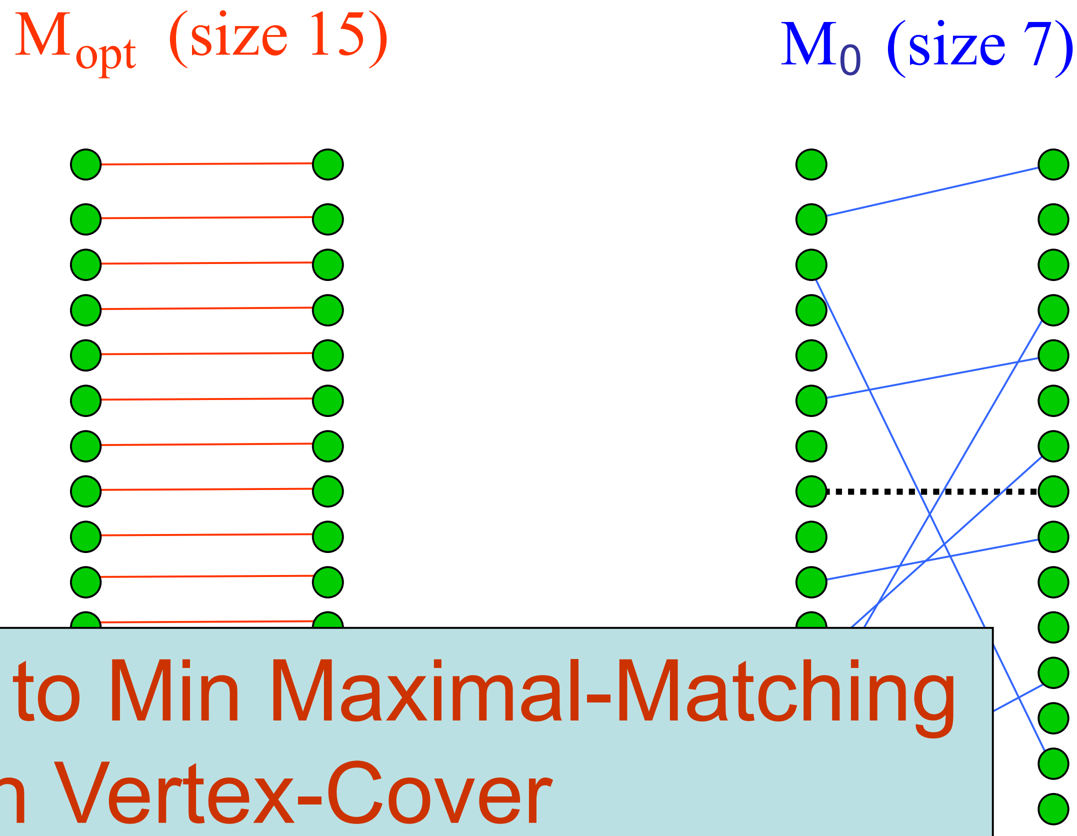
Local Search

1:	(a b)	a:	1 2 3
2:	b (c a)	b:	2
3:	b	c:	3 (1 2)

- Arbitrary tiebreak and Gale-Shapley $\Rightarrow M_0$ (guaranteed stability for the original instance)
- Size=size+1 by *INCREASE* $\Rightarrow M_1$
- $\Rightarrow M_2 \Rightarrow \cdots \Rightarrow M_k$
- *INCREASE* fails and output M_k

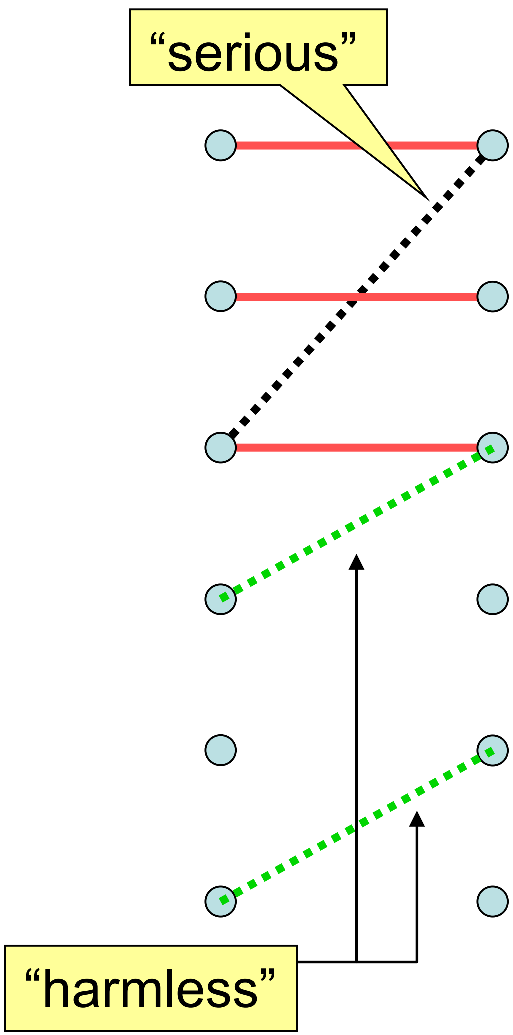
Ratio 2.0 is Trivial

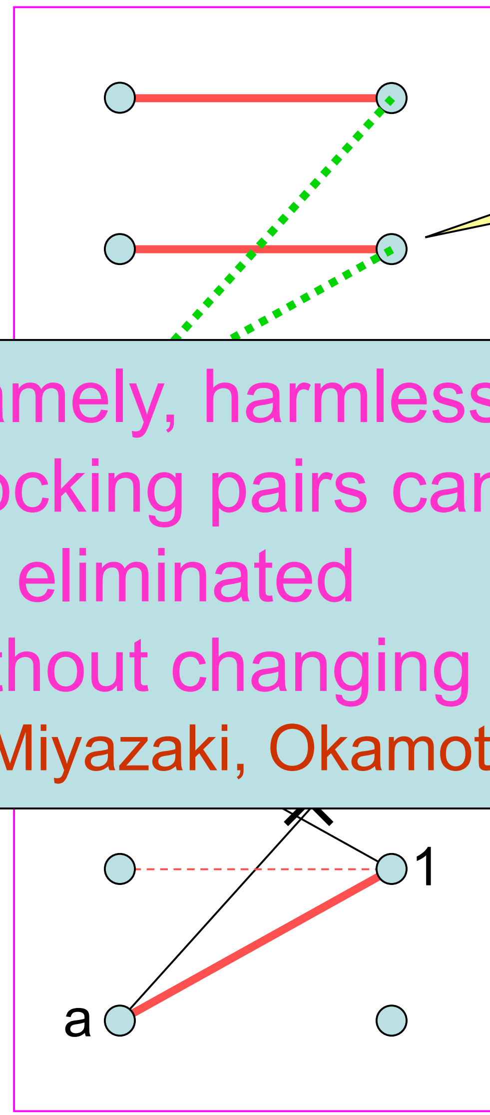
The first M_0 is already not too small



Similar to Min Maximal-Matching and Min Vertex-Cover

Harmless Blocking Pairs





a’s most favorite*

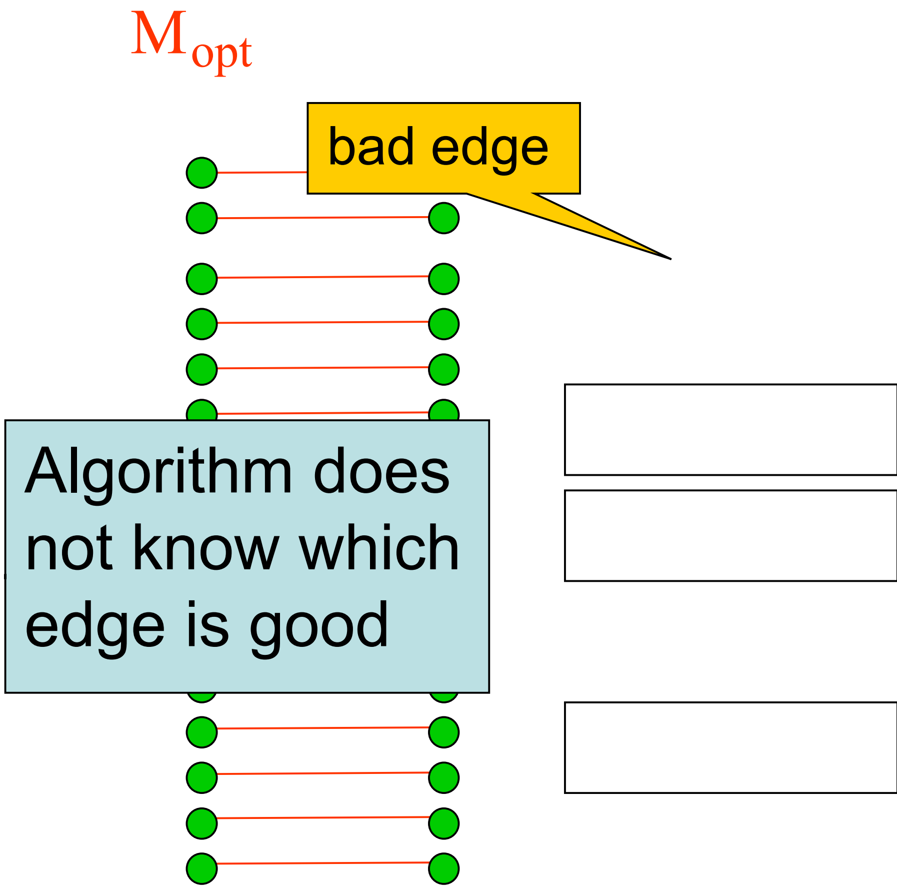
Namely, harmless blocking pairs can be eliminated without changing size [I, Miyazaki, Okamoto 04]

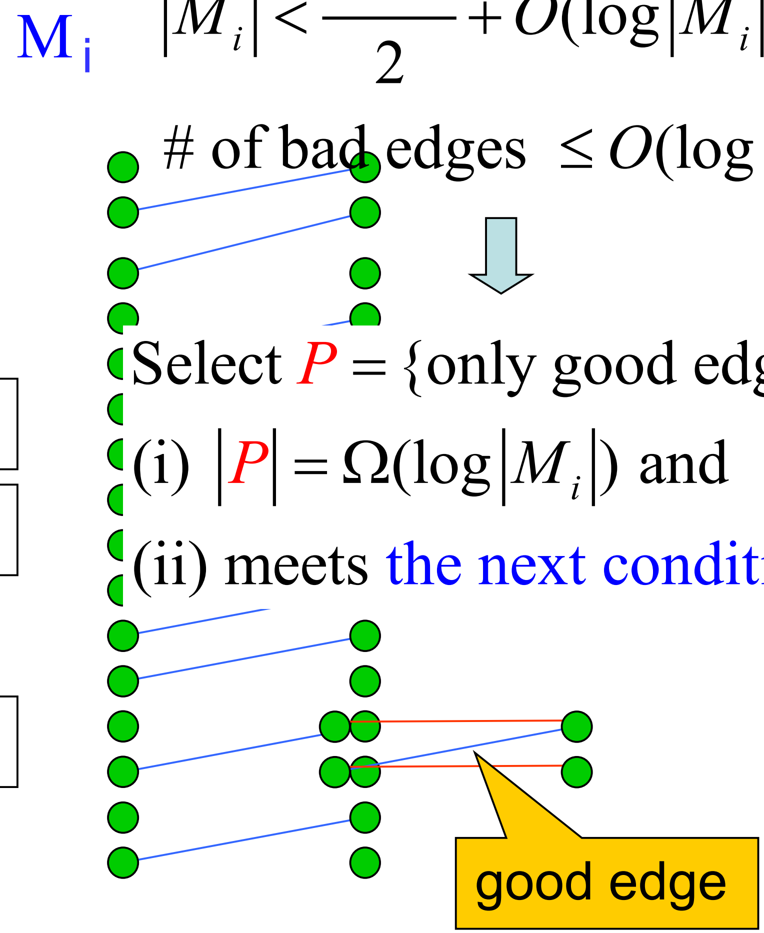
ze does not
ange
new serious
ocking pairs
BPs) Why?

Because of *

Because 1 gets happier

INCREASE: Basic Ideas





$$|M_i| < \frac{OPT}{2} + O(\log|M_i|) \Rightarrow$$

$$\#\text{ of bad edges } \le O(\log|M_i|)$$

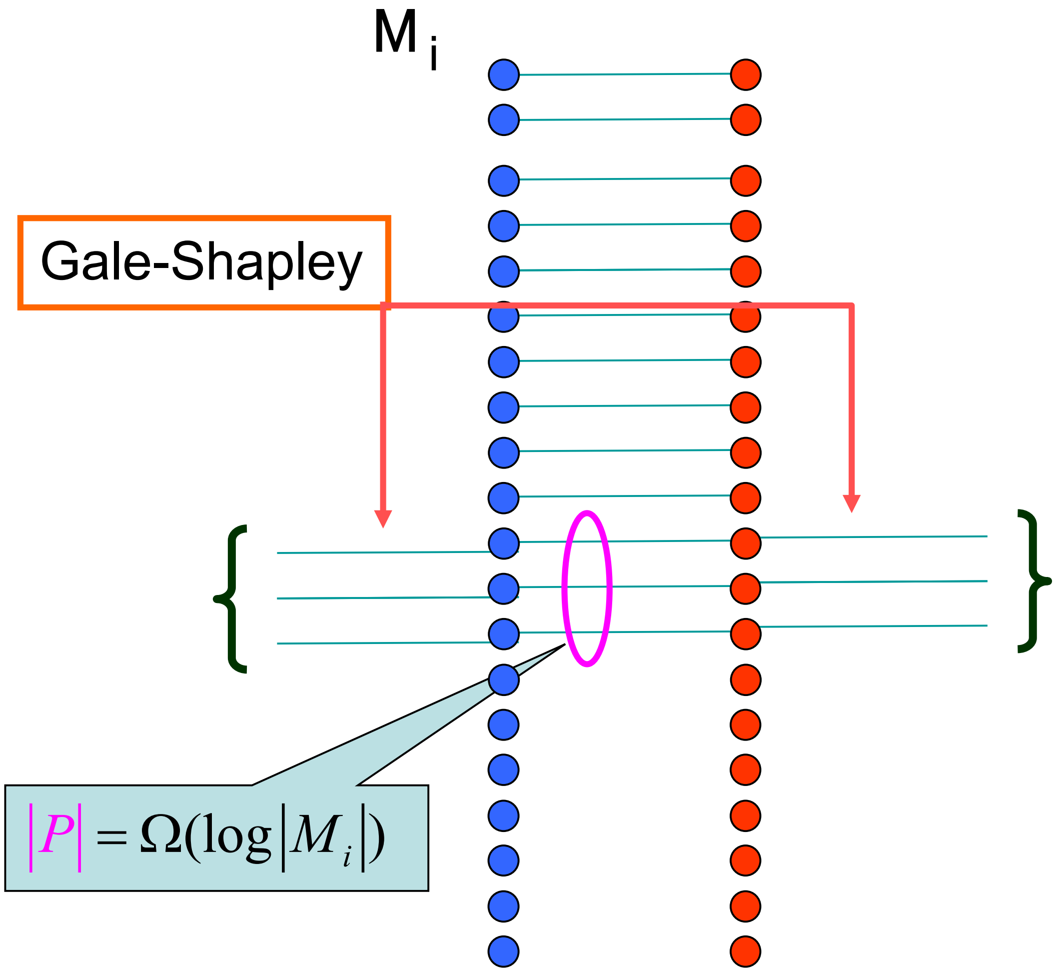
⇩

Select P = {only good edges} s.t.

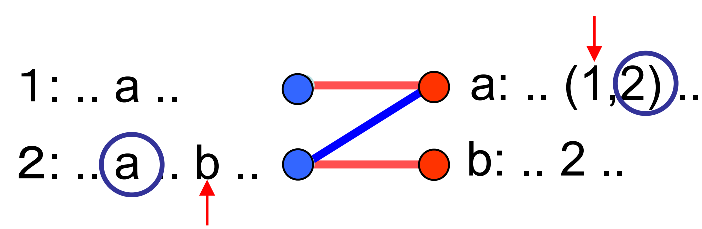
(i) $|P| = \Omega(\log|M_i|)$ and

(ii) meets the next condition

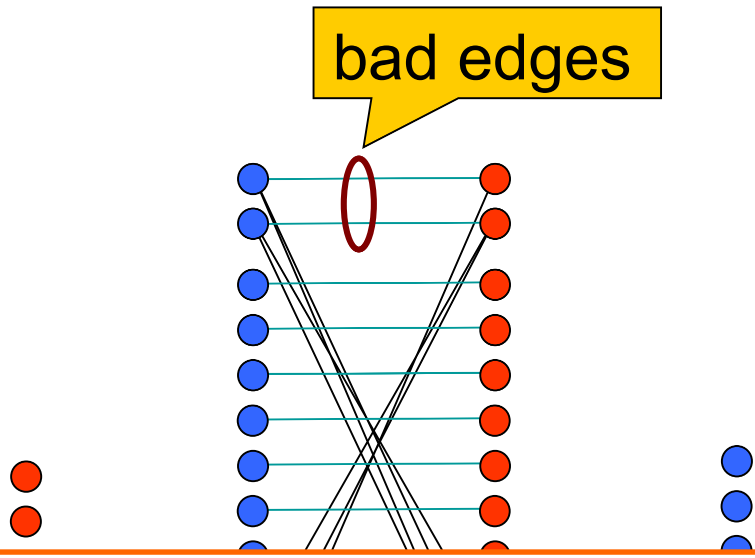
What Kind of Condition?



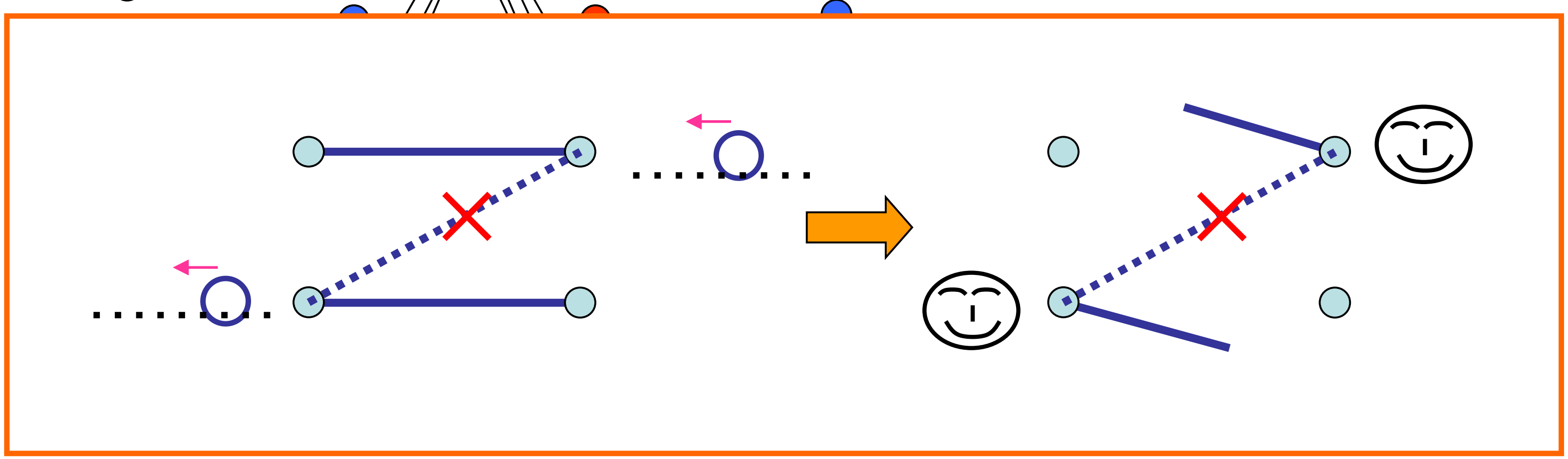
New partners are as good as OPT



Why the Condition is Desirable?



These blocking pairs may be possible but no others



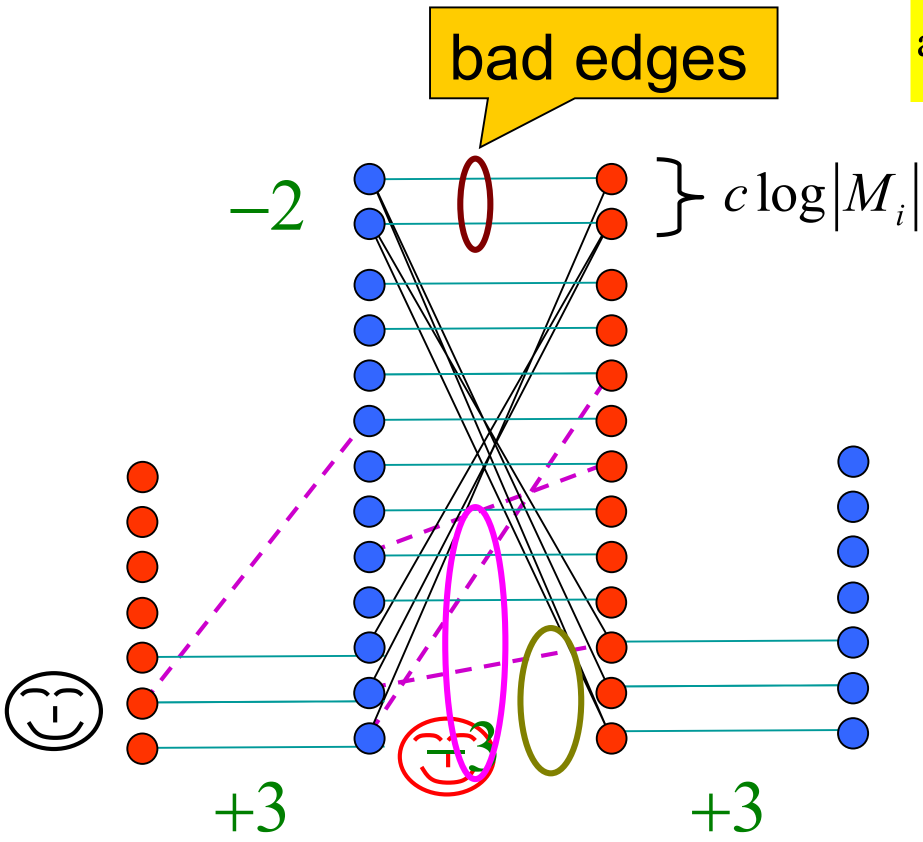
Why the Conditio… le?

$|M_i| < \frac{OPT}{2} + O(\log|M_i|) \Rightarrow$

# of bad edges $\le O(\log|M_i|) \Rightarrow$

$INCREASE$ always succeeds $\Rightarrow$

approx ratio $\le 2 - c\frac{\log n}{n}$



set $|P| = c\log|M_i| + 1$ then we can increase the size by one

How to find such a P?

Lemma For any Q s.t.

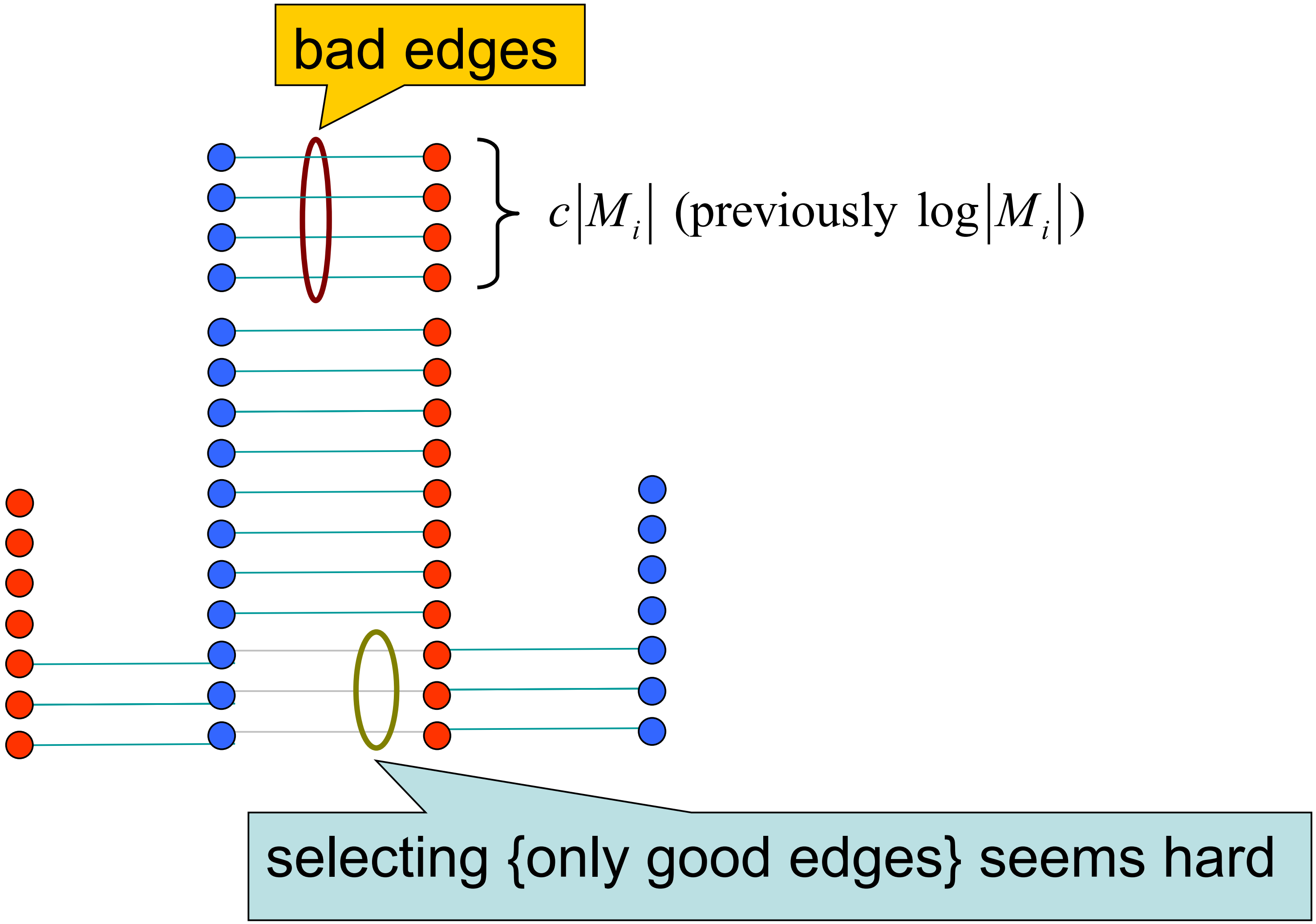
(i) $|Q| = 4|P|$ and

(ii) $Q = \{\text{all good pairs}\}$,

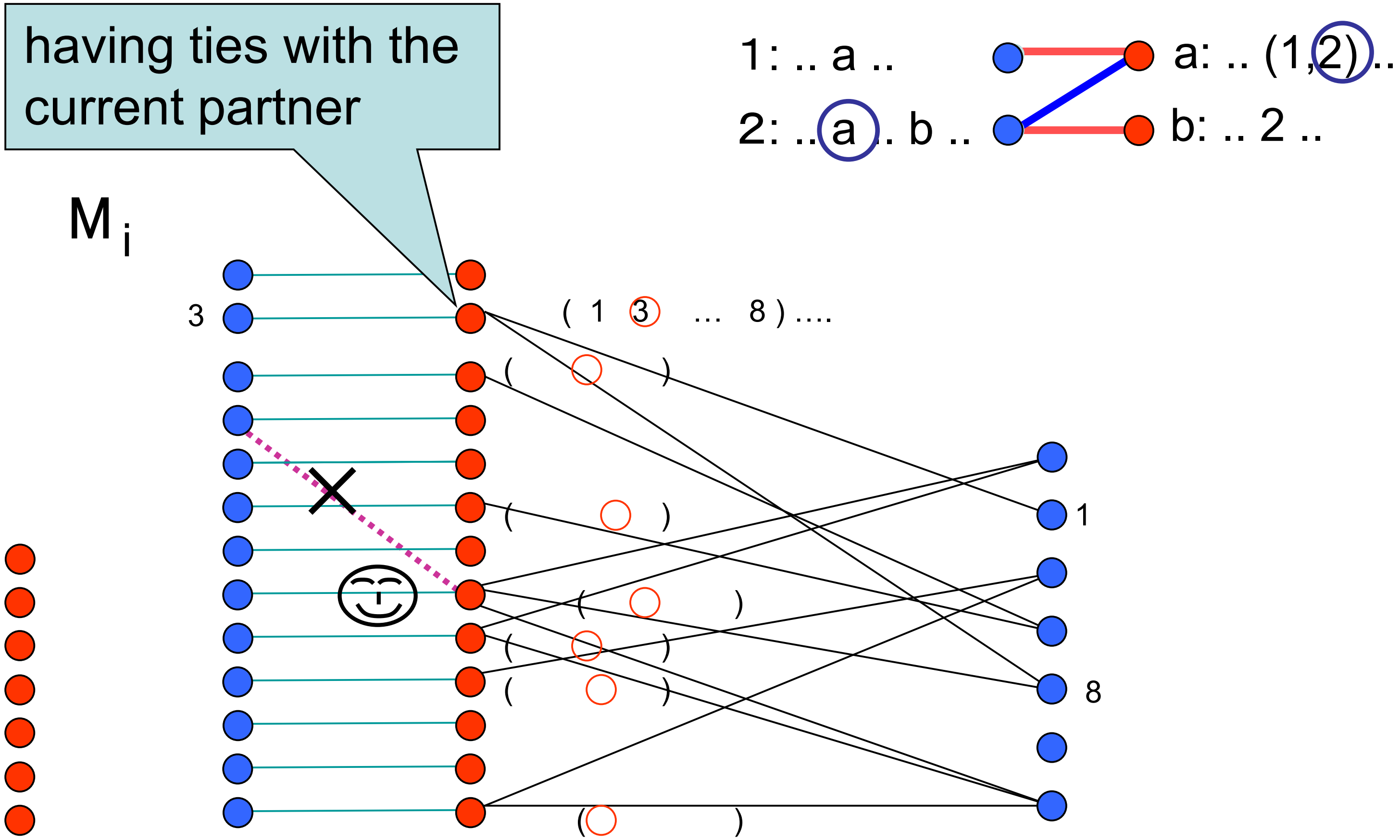
$\exists$ such a $P \subseteq Q$

New *INCREASE* Achieving 1.875

[I, Miyazaki, Yamauchi 07]

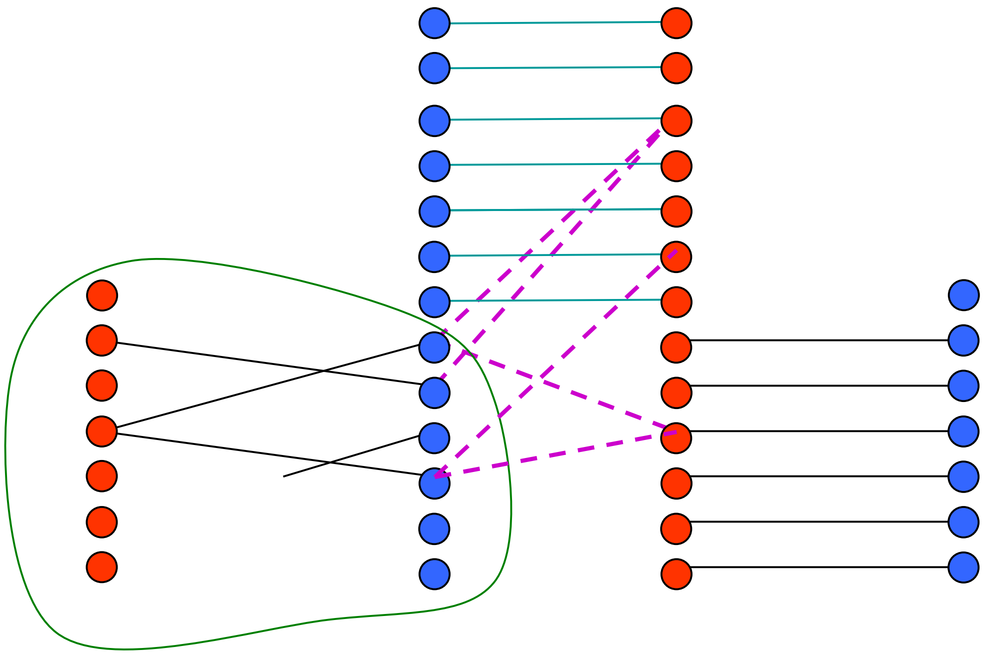


Changing Pairs

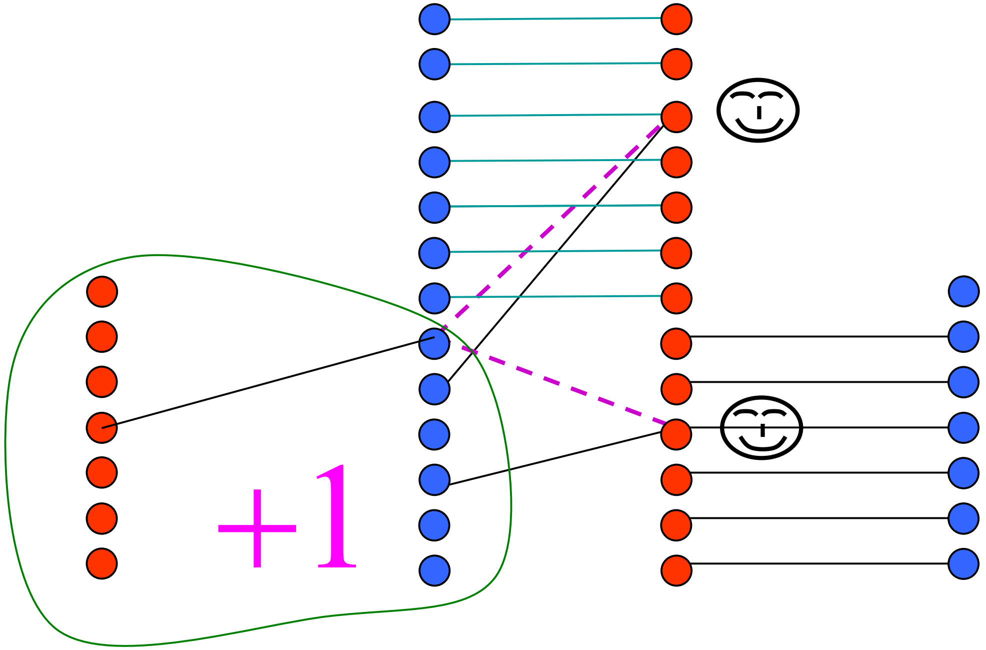


so far the same size

Trying to Get +1



How to Get +1

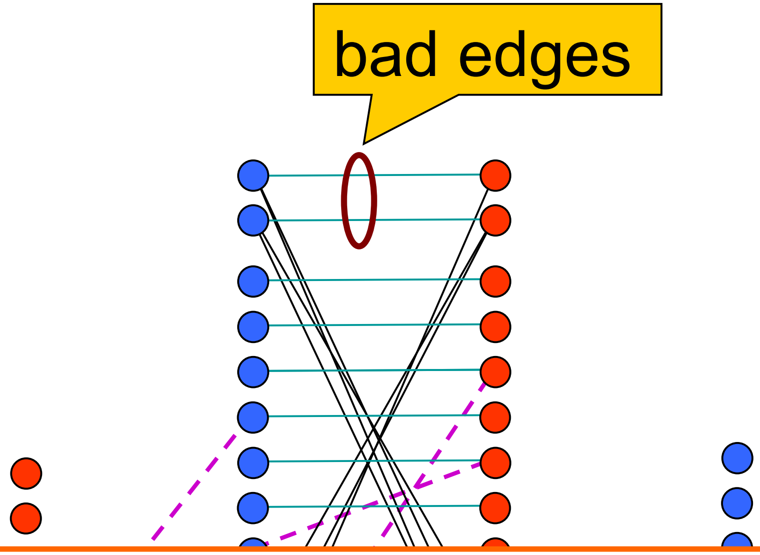


Some Open Problems

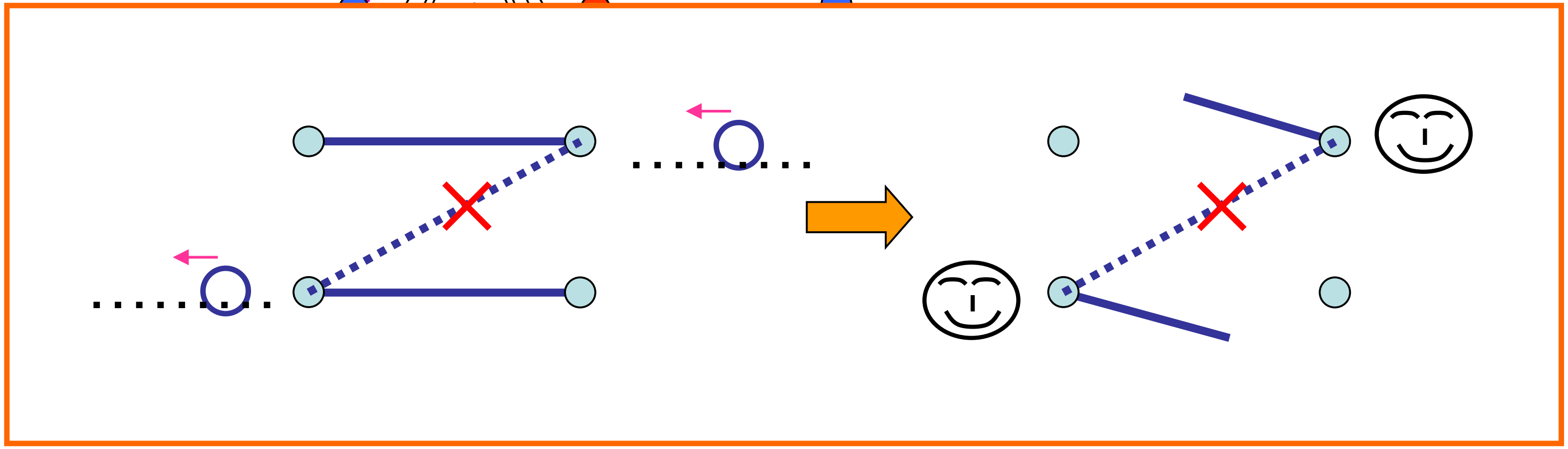
- Approximability of MAX SMTI
- SM is in NC or P-complete
- The maximum possible number of stable matchings (experiments and conjectures)
- New problems are constantly coming
- Game theoretic approaches

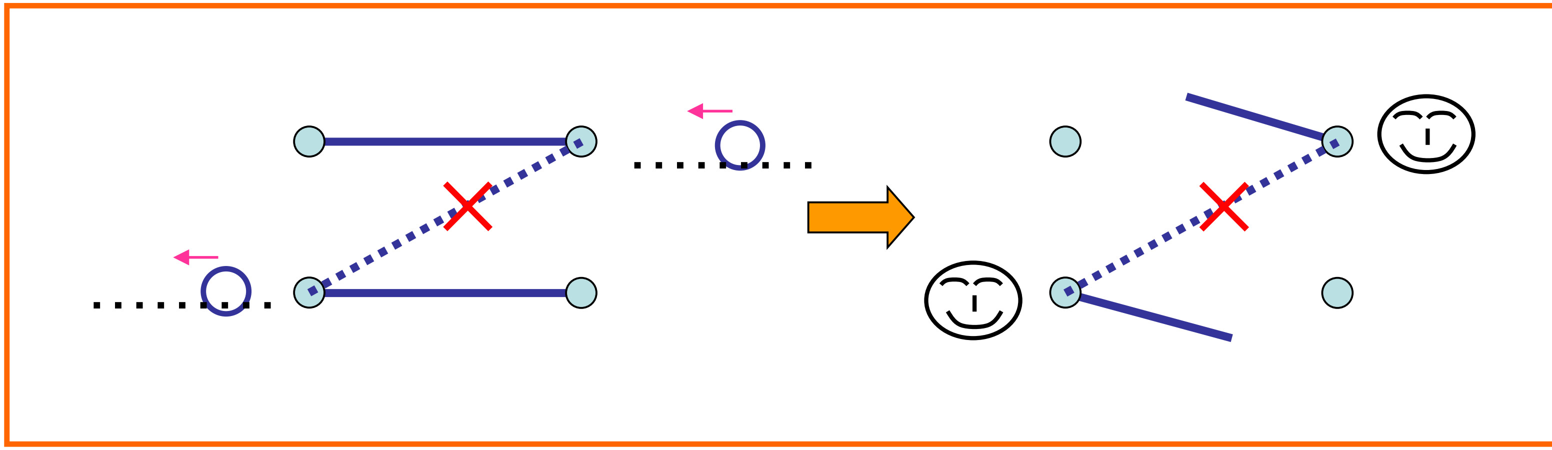
Thank you

Why the Condition is Desirable?

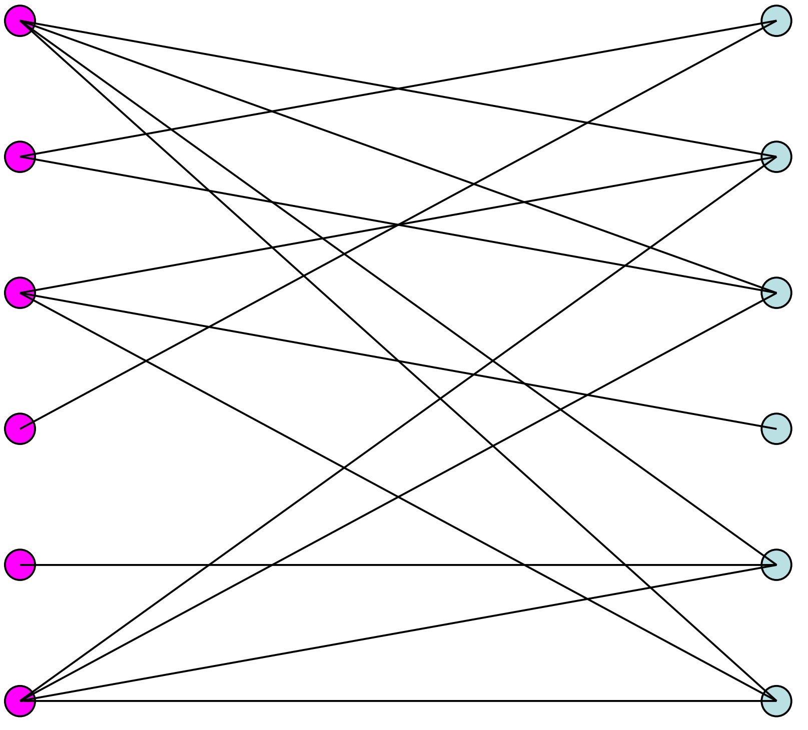


These blocking pairs may be possible but no others

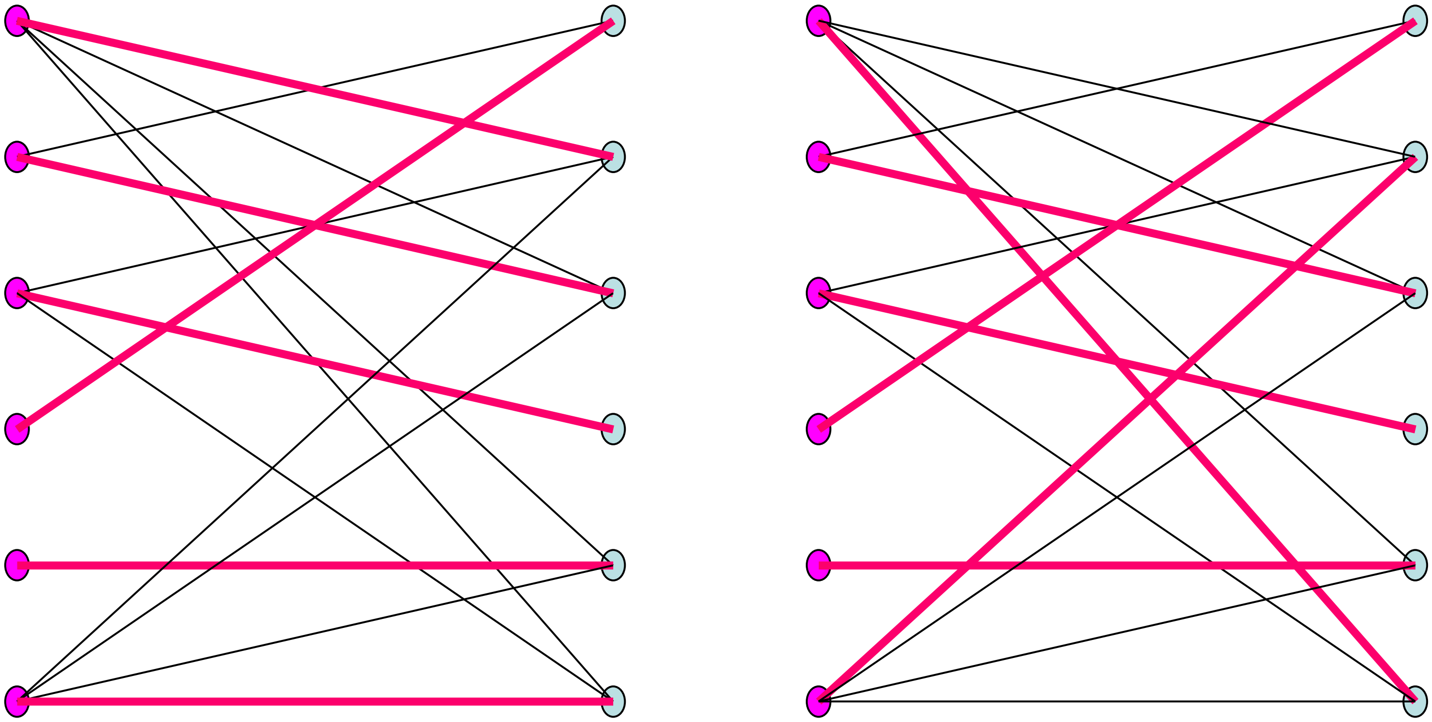




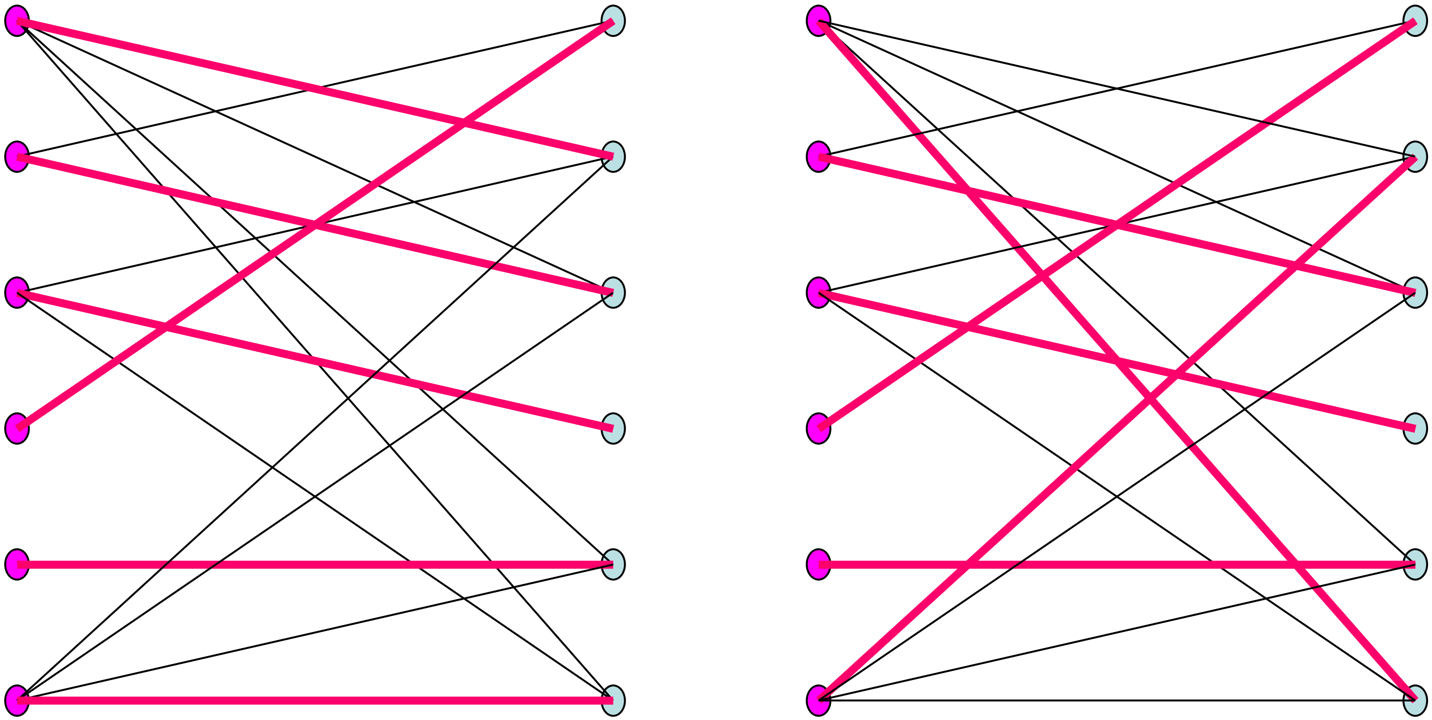
Bipartite Matching



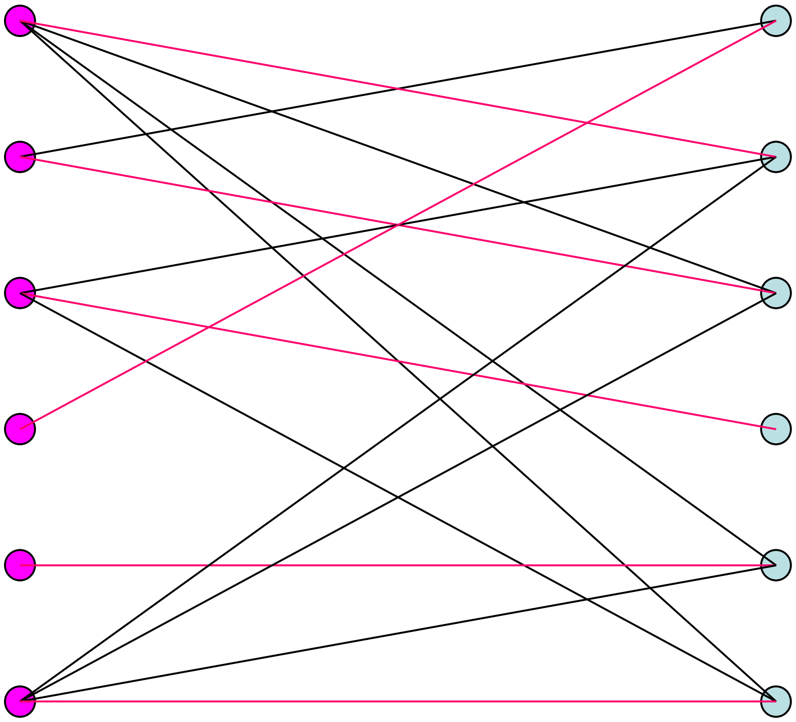
Bipartite Matching



Bipartite Matching



Bipartite Matching

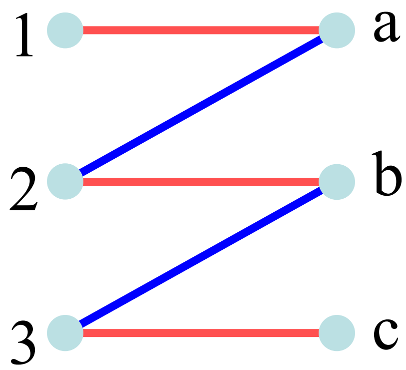


Cycles and Paths

1: … a …

2: … b … a …

3: … c … b …

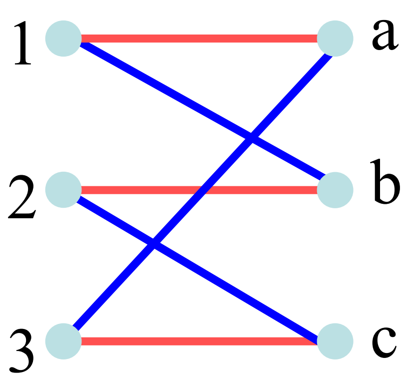


a: … 2 … 1 …

b: … 3 … 2 …

c: … 3 …

This cannot happen!

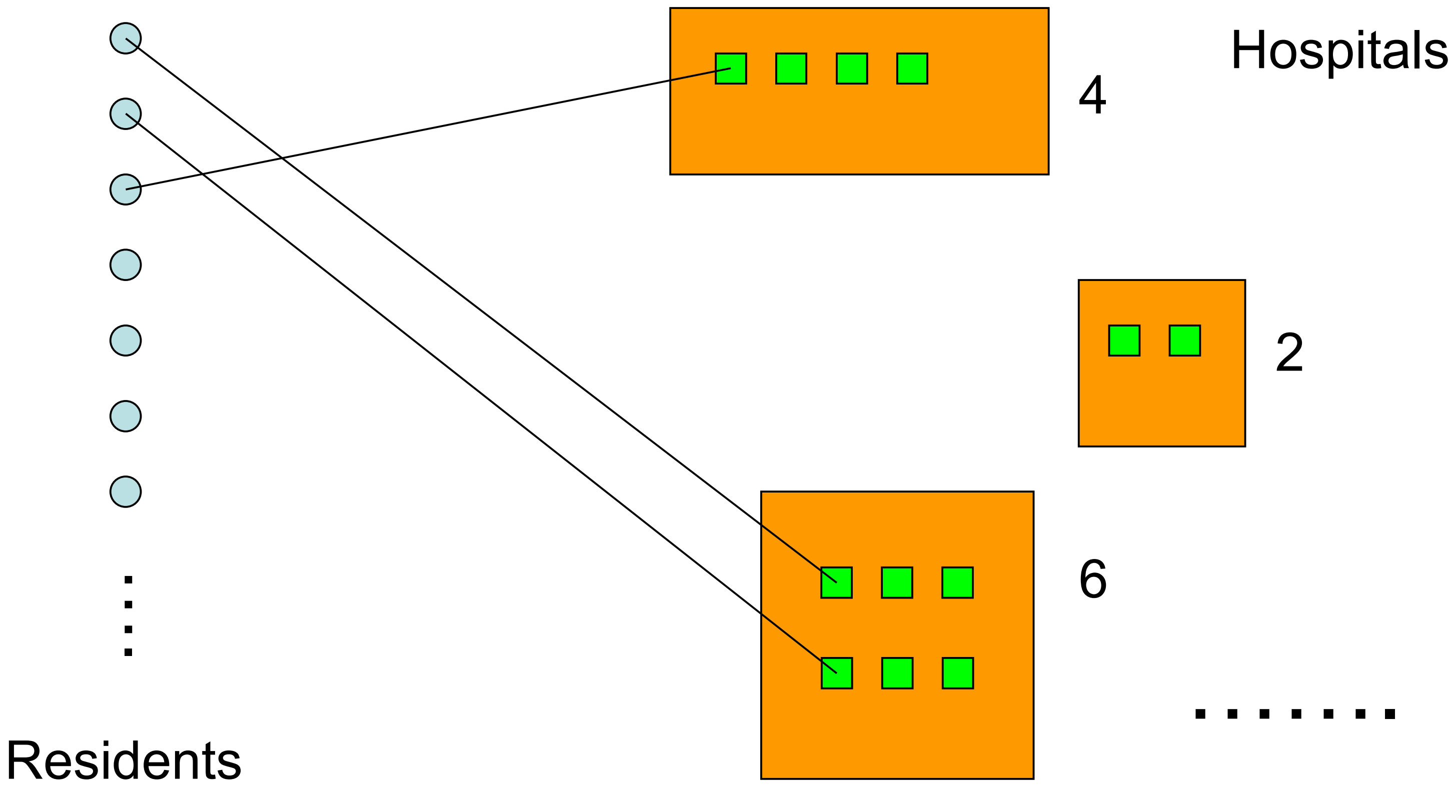


Only paths are possible

Stable Matching

1:	a	c	b	d	e	a:	2	1	3	4	5
2:	c	a	e	b	d	b:	2	1	4	5	3
3:	b	a	e	d	c	c:	1	2	3	5	4
4:	c	b	d	e	a	d:	3	1	4	2	5
5:	c	d	b	e	a	e:	4	3	1	2	5

One-to-Many Matchings



Operations for Stable Matchings

Is a stable matching unique? No.

1:	a	c	**b**	d	a:	**2**	4	1	3
2:	b	d	c	**a**	b:	3	**1**	2	4
3:	c	a	**d**	b	c:	**4**	2	3	1
4:	d	b	a	**c**	d:	1	**3**	4	2

M_1

1:	a	c	**b**	d	a:	2	**4**	1	3
2:	b	d	**c**	a	b:	3	**1**	2	4
3:	c	a	**d**	b	c:	4	**2**	3	1
4:	d	b	**a**	c	d:	1	**3**	4	2

$M_1 \wedge M_2$

1:	a	c	b	**d**	a:	2	**4**	1	3
2:	b	d	**c**	a	b:	**3**	1	2	4
3:	c	a	d	**b**	c:	4	**2**	3	1
4:	d	b	**a**	c	d:	**1**	3	4	2

M_2

1:	a	c	b	**d**	a:	**2**	4	1	3
2:	b	d	c	**a**	b:	**3**	1	2	4
3:	c	a	d	**b**	c:	**4**	2	3	1
4:	d	b	a	**c**	d:	**1**	3	4	2

$M_1 \vee M_2$

Extensions

- Stable roommate problem ⇦
- Relaxation of preference lists ⇦
- Different definitions for stability
- One-many matching
- Popular matching
- Game-theoretic approaches
- ……