Mathematical Program with Equilibrium Constraints (MPEC)

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MPEC model

$$\begin{aligned}
\min_{z} \quad & f(z) & \\
\text{s.t.} \quad & g(z) \geq 0, & \text{(side inequality)} \\
& h(z) = 0, & \text{(side equality)} \\
& 0 \leq r(z) \perp s(z) \geq 0 & \text{(complementarity constraint)}
\end{aligned}$$

$g : \mathbb{R}^n \to \mathbb{R}^{m_g}$, $h : \mathbb{R}^n \to \mathbb{R}^{m_h}$, $r, s : \mathbb{R}^n \to \mathbb{R}^m$. $\perp$ denotes perpendicularity.

◦ also called the Mathematical Program with Complementarity Constraints (MPCC).

Outline

Complementarity Problem (CP)

$$0 \leq r(z) \perp s(z) \geq 0$$

- **LCP:** both sides of $\perp$ are linear functions
- **NCP:** not both sides of $\perp$ are linear

Mixed complementarity problem:

$$0 \leq r_j(z) \perp s_j(z) \geq 0, \forall j \in J \text{ (an index set)}$$
$$\text{free } r_j(z) \perp s_j(z) = 0, \forall j \notin J$$

Variational inequality

Variational Inequality (VI): Given a set K and a function $F : K \to \mathbb{R}^n$, $VI(K, F) :=$

$$(y - x)^T F(x) \geq 0, \forall y \in K.$$

$\dashrightarrow$ Solve for an x satisfying the above condition.

- **AVI:** If F is affine and K is polyhedral.

Relationship between CP and VI

⋆ CP can be obtained by specializing VI. VI is more general than CP.

⋆ When the set K is a cone, VI can be written as CP.

⋆ VI is a nontrivial extension of a nonlinear program (NLP).

From NLP to VI/CP

Constrained optimization program:

$$\begin{aligned} \min \quad & \theta(x) \\ \text{s.t.} \quad & x \in K \end{aligned}$$

∘ If K is convex, a local minimum x^* satisfies

$$(x - x^*)^T \nabla\theta(x) \geq 0, \; \forall x \in K.$$

$\dashrightarrow VI(K, \nabla\theta)$.

∘ `Known result:` At what condition a function, F is the gradient of another function θ, i.e., F is integrable?

— The Jacobian matrix $JF(x)$ is symmetric $\forall x \in$ feasible region.

From VI to NLP

Consider the set K:

$$K \equiv \{x \in R^n : h(x) = 0,\ g(x) \leq 0\}$$

$h : R^n \to R^\ell$, $g : R^n \to R^m$.

If x solves VI(K,F), then x solves the following NLP

$$\begin{aligned} \min \quad & y^T F(x) \\ \text{s.t.} \quad & y \in K \end{aligned}$$

, i.e., $y^* = x$.

KKT condition

The above NLP has the optimality (necessary) condition:

∘ Assume CQ holds at x, then there exist vectors $\mu \in R^\ell$ and $\lambda \in R^m$ such that

$$0 = F(x) + \sum_{j=1}^{\ell} \mu_j \nabla h_j(x) + \sum_{i=1}^{m} \nabla g_i(x)$$

$$0 = h(x)$$

$$0 \leq \lambda \perp -g(x) \geq 0.$$

⇢ A mixed CP in x, μ, λ.

Noncooperative game

A noncooperative game, N players.
∘ Player i's strategy set: K_i
∘ Player i''s strategy: x^i
∘ Player i's cost function: $\theta_i(\mathbf{x})$, depends on all players' strategies. $\mathbf{x}$ consists of all subvectors x^i

Noncooperative game/Cooperative game

Given other $N-1$ players' strategies $\mathbf{x}^{-i}$, player i's optimization problem:

$$\begin{aligned} \min \quad & \theta_i(x^i, \mathbf{x}^{-i}) \\ \text{s.t.} \quad & x^i \in K_i. \end{aligned}$$

Nash Equilibrium

The Nash equilibrium is N players' strategies such that no player has the incentive to unilaterally deviate from the current strategy.

Consider convex K_i and convex cost function θ_i.

$\mathbf{x}$ is a Nash equilibrium if and only if $\forall$ individual i,

$$(y^i - x^i)^T \nabla_{x^i} \theta_i(\mathbf{x}) \geq 0, \ \forall y^i \in K_i$$

`Special case:` two-person zero-sum game

$N = 2$, $\theta_1(x) = -\theta_2(x)$.

Concatenating individual VIs

Concatenating the gradients

$$\mathbf{F}(\mathbf{x}) \equiv (\nabla_{x^i}\theta_i(\mathbf{x}))_{i=1}^N.$$

and form the Cartesian product

$$\mathbf{K} \equiv \prod_{i=1}^N K_i$$

As a result, $\mathbf{x}$ is a Nash equilibrium iff $\mathbf{x} \equiv (x^i)_{i=1}^N$ solves the VI

$$(\mathbf{y}-\mathbf{x})^T\mathbf{F}(\mathbf{x}) \geq 0, \forall \mathbf{y} \in \mathbf{K}$$

Mixed CP expression of Nash equilibrium

- Suppose set K_i is specified by inequalities and equalities.
- concatenating N KKT systems

$\dashrightarrow$ a mixed CP expression of the Nash Equilibrium.

Bimatrix game

Bimatrix game: $\Gamma(A, B)$

⋆ Player I and player II participate.
⋆ A and B are **costs matrices** incurred by players I and II respectively.
⋆ Suppose player I has m strategies and player II has n strategies. $A, B \in R^{m \times n}$.
⋆ Two strategy settings: pure strategy, mixed strategy

Strategy

1. **Pure strategies:** When player I chooses strategy i and player II chooses strategy j, player I incurs cost A_{ij} and player II incurs B_{ij}.

2. **Mixed strategies:** Introduce $x \in R^m$ and $y \in R^n$ the probabilities of choosing each strategy for player I and II respectively. Player I incurs *expected cost* $x^T Ay$ and player II incurs $x^T By$.

Bimatrix game—equilibrium of mixed strategies

A pair of mixed strategies (x^*, y^*) is said to be a **Nash equilibrium** if

$$(x^*)^T A y^* \leq x^T A y^*, \forall x \geq 0 \text{ and } \sum_{i=1}^{m} x_i = 1$$

$$(x^*)^T B y^* \leq {x^*}^T A y, \forall y \geq 0 \text{ and } \sum_{j=1}^{n} y_i = 1$$

Supply-Demand market equilibrium

1. Supply side:

$$
\begin{aligned}
\min \quad & c^T x \\
\text{s.t.} \quad & Ax \geq b \\
& Bx \geq r^* \\
& x \geq 0.
\end{aligned}
$$

c: cost for the supply activities

x: production activity level

$Ax \geq b$: technological constraints

$Bx \geq r^*$: demand requirement constraint

Supply-Demand market equilibrium

2. Demand side:

$$r^* = Q(p^*) = Dp^* + d$$

$Q(\cdot)$: market demand function, assumed affine

p^*: prices

r^*: demand quantities

Supply-Demand market equilibrium

Denote π^*: the shadow price (i.e., the market supply prices) corresponding to the constraint $Bx \geq r^*$

3. Equilibrating condition:

$$p^* = \pi^*$$

Mathematically, we are to find p^* and r^* so that the above 3 things are satisfied.

Supply-Demand market equilibrium—LCP formulation

First we write the optimality condition for supply side, where v is the multiplier corresponding to $Ax \geq b$:

$$
\begin{aligned}
0 \leq c - A^T v^* - B^T \pi^* \quad &\perp \quad x^* \geq 0 \\
0 \leq Ax^* - b \quad &\perp \quad v^* \geq 0 \\
0 \leq Bx^* - r^* \quad &\perp \quad \pi^* \geq 0
\end{aligned}
$$

Then substitute r^* by $Dp^* + d$ and π^* by p^*. Done.

Cournot production problem

Cournot production problem: The price of goods depends on total quantity in the market.

◦ quantity competition. (Cournot competition)

◦ a phenomenon accompanying with oligopoly

◦ may be reduced to monopoly or extend to perfect competition

Cournot production and distribution problem formulation

Plants are on a network with node set $\mathcal{N}$ and arc set $\mathcal{A}$.
M producers. $\mathcal{N}_f$ are markets where firm f has a plant.

Variables:

x_{fa} : amount of flow controlled by f on link a
s_{fi} : amount of the commodity produced by firm f at node i
d_{fj} : amount of the commodity delivered by firm f to node j

Parameters:

$C_{fi}(s_{fi})$: cost to f of producing s_{fi} units of the commodity at i.
CAP_{fi} : capacity of firm f to produce at i.
$c_{fa}(x_{fa})$: cost to f of shipping x_{fa} units on link a.

Total quantity decides price

Denote

$$Q_j = \sum_{f=1}^{M} d_{fj}$$

The unit price is expressed as

$$p_j(Q_j)$$

Firm f's profit maximization:

$$\max \theta_f \equiv \sum_{j \in \mathcal{N}} d_{fj} p_j(Q_j) - \sum_{i \in \mathcal{N}_f} C_{fi}(s_{fi}) - \sum_{a \in A} x_{fa} c_a(x_{fa})$$

Optimal production and distribution

Constraints:

$$s_{fi} \leq CAP_{fi}, \qquad \forall i \in N_f$$

$$d_{fi} + \sum_{a \in A_i^+} x_{fa} = s_{fi} + \sum_{a \in A_i^-} x_{fa}, \qquad \forall i \in \mathcal{N}_f$$

$$d_{fi} + \sum_{a \in A_i^+} x_{fa} = \sum_{a \in A_i^-} x_{fa}, \qquad \forall i \in \mathcal{N} \backslash \mathcal{N}_f$$

$$d_{fi}, s_{fi}, x_{fa} \geq 0.$$

A_i^+ : set of arcs with i as the beginning node

A_i^- : set of arcs with i as the ending node.

⋆ Denote K_f the constraints set containing above constraints.

Nash-Cournot Equilibrium

Let x^f be the stack of variables $[d, s, x]$ and $\mathbf{x} = (x^f)_{f=1}^M$.

⋆ Rewrite firm f's optimization problem.

Standard form:

$$\begin{aligned} \max \quad & \theta_f(\mathbf{x}) \\ \text{s.t.} \quad & x^f \in K_f. \end{aligned}$$

Nash-Cournot equilibrium

$\mathbf{x}$ is a equilibrium iff $\mathbf{x} \equiv (x^i)_{i=1}^N$ solves

$$(\mathbf{y} - \mathbf{x})^T \mathbf{F}(\mathbf{x}) \geq 0, \forall \mathbf{y} \in \mathbf{K}$$

where $\mathbf{F}(\mathbf{x})$ be the concatenation of $-\nabla_{x^f} \theta_f(\mathbf{x})$.

Cournot production and distribution—electricity network

Electricity: an oligopoly market — Cournot production

Main aspects:

◦ generation **[multiple generation plants on a node]**

◦ transmission **[extra supply through arcs]**

◦ distribution **[fulfilling demand on each node]**

Electricity market notations

Additional notations needed:

Parameters:

G_{fi} : set of generation plants owned by firm f at node $i \in N_f$

CAP_{fih} : generation capacity at plant $h \in G_{fi}$

CAP_a : transmission capacity on link a

$\rho_a(\mathbf{z})$: transmission price on link a depending on total flow $\mathbf{z}$

C_{fih} : the cost of generation to firm f at site i and plant h.

Variables:

y_{fih} : amount produced at plant $h \in G_{fi}$

Electricity market notations

Firm f's profit maximization problem:

$$\begin{aligned} \max \quad & \theta_f(x^f) \\ \text{s.t.} \quad & x^f \in K_f(\mathbf{x}^{-f}) \end{aligned}$$

x^f : firm f's decision variables including $[d, y, x]$

$\mathbf{x}^{-f}$: decisions made by firms other than f

$\theta_f(x^f)$: profit of firm f resulting from the decision

$K_f(\mathbf{x}^{-f})$: constraints set of firm f, where some of the parameters are determined by other firms.

Electricity market notations

$$\theta_f(x^f) = \sum_{j \in N} d_{fj} p_j \left(\sum_{g=1}^{M} d_{gj} \right) - \sum_{i \in N_f} \sum_{h \in G_{fi}} C_{fih}(y_{fih}) - \sum_{a \in A} x_{fa} \rho_a$$

$$K_f(\mathbf{x}^{-f}) \equiv$$

$$\begin{aligned}
& y_{fih} \le CAP_{fih}, \forall h \in G_{fk}, && \forall i \in N_f \\
& d_{fi} + \sum_{a \in A_i^+} x_{fa} = \sum_{h \in G_{fi}} y_{fih} + \sum_{a \in A_i^-} x_{fa}, && \forall i \in N_f \\
& d_{fi} + \sum_{a \in A_i^+} x_{fa} = \sum_{a \in A_i^-} x_{fa}, && \forall i \in N \backslash N_f \\
& \sum_{f' \in F} x_{f'a} \le CAP_a, && \forall a \in A \text{ [link capacity]} \\
& x_{fa}, y_{fih}, d_{fi} \ge 0.
\end{aligned}$$

Electricity market notations

Further define $\tilde{K}_f \equiv$

$$
\begin{aligned}
& y_{fih} \leq CAP_{fih}, \forall h \in G_{fk}, && \forall i \in N_f \\
& d_{fi} + \sum_{a \in A_i^+} x_{fa} = \sum_{h \in G_{fi}} y_{fih} + \sum_{a \in A_i^-} x_{fa}, && \forall i \in N_f \\
& d_{fi} + \sum_{a \in A_i^+} x_{fa} = \sum_{a \in A_i^-} x_{fa}, && \forall i \in N \backslash N_f
\end{aligned}
$$

Define

$$K_f(\mathbf{x}) \equiv \{x^f : \sum_{f' \in F} x_{f'a} \leq CAP_a\} \cap \tilde{K}_f$$

$$\Omega \equiv \{\mathbf{x} : \text{ all } \mathbf{x} \text{ satisfying link capacity}\}$$

$$\mathbf{K} \equiv \left(\prod_{f=a}^{M} \tilde{K}_f \right) \cap \Omega$$

Electricity market notations

$$\mathbf{F}(\mathbf{d},\mathbf{y},\mathbf{z}) = \begin{pmatrix} -\dfrac{\partial \theta_f(\mathbf{x})}{\partial d_{fi}} : \forall f,i \\ \dfrac{dC_{fih}(y_{fih})}{dy_{fih}} : \forall f,i,h \\ \rho_{\cdot}(\mathbf{z}) \otimes e \end{pmatrix}$$

$\rho.$ is the vector of all ρ_a, $a \in A$, $\otimes$ denotes the Kronecker product, and e is the vector of ones in R^M. This produces M copies of $\rho.$.

$\dashrightarrow$ Equilibrium $(\mathbf{d},\mathbf{y},\mathbf{z})$ are those solve the $VI(\mathbf{K},\mathbf{F})$.

Structural Estimation

The structural estimation is a (relatively) new-born technique which involves:

1. Assuming a parametric model for the system
   - including probabilistic assumptions on random quantities
2. Deducing a set of necessary (structural) equations for unknown parameters
   - including optimality condition of optimization within the system
3. Solving an MPEC corresponding to a generalized method of moment (GMM) estimate of the parameters.
   - including optimality condition of optimization within the system and the orthogonality conditions of instrumental variables used in GMM

Structural Estimation: pure characteristics

To describe the demand of consumers,

- ▷ Discrete choice, 1974
- ▷ Random Coefficients Logit (or BLP model), 1995
- ▷ Pure Characteristics (or PCM), 2007

In PCM,
the utility for consumer i buying product j in market t is

$$u_{ijt} = \mathbf{x_{jt}^T}\boldsymbol{\beta_i} - \alpha_i p_{jt} + \xi_{jt},$$

$\mathbf{x_{jt}} \in R^K$: observed product characteristics,

p_{jt} : price of product j in market t,

$\boldsymbol{\beta_i} \in R^K$ and α_i : consumer specific coefficients, and

ξ_{jt} : the only unobserved characteristic.

Structural Estimation: pure characteristics

Select the coefficients $\boldsymbol{\beta_i}$, α_i and ξ_{jt} so the utility is appropriate.

Structures include market-level observations that should be met:

⋆ Market share (or product quantity sold)

⋆ Distribution of the random coefficients $\boldsymbol{\beta_i}$ and α_i

⋆ Observed product price

⋆ Distribution of the marginal cost

⋆ Competitive environment $\dashrightarrow$ a Game with $F+1$ players

Model development

- Introduce π_{ijt}: probability for consumer i to buy product j in market t.

Rational consumers do the following:

$$0 \leq \pi_{ijt} \quad \perp \quad \gamma_{it} - [\mathbf{x_{jt}^T}\boldsymbol{\beta_i} - \alpha_i p_{jt} + \xi_{jt}] \geq 0$$

$$0 \leq \gamma_{it} \quad \perp \quad \pi_{i0t} = 1 - \sum_{j=1}^{J} \pi_{ijt} \geq 0,$$

where $\gamma_{it} = \max\left\{ 0, \max_{1 \leq \ell \leq J} \left(\mathbf{x_{\ell t}^T}\boldsymbol{\beta_i} - \alpha_i p_{\ell t} + \xi_{\ell t} \right) \right\}.$

Model development

- The $F + 1$ players in the Game are F firms and a virtual league of consumers.
  ⋆ F firms: pricing problem
  ⋆ The league of consumers: maximizing the aggregated utility, also called "market optimization" problem.
- Use Generalized Method of Moments (GMM) for minimizing residuals.

The estimation model

$\mathbb{QPNCC}_{EsP,NB}(\mathbf{Z}_\xi;\ \boldsymbol{\Lambda}_\xi;\ \mathbf{Z}_\omega;\ \boldsymbol{\Lambda}_\omega;\ \mathcal{M}_t;\ N;\ q;\ p^{obs};\ x;\ y;\ \eta;\ w;)$:

$$\min_{\theta \in \Upsilon;\, mc;\, \xi;\, \omega;\, \mathbf{z}} \quad \tfrac{1}{2}\,\boldsymbol{\xi}^{\boldsymbol{T}}\mathbf{Z}_\xi\boldsymbol{\Lambda}_\xi\mathbf{Z}_\xi^{\mathbf{T}}\boldsymbol{\xi} + \tfrac{1}{2}\,\boldsymbol{\omega}^{\boldsymbol{T}}\mathbf{Z}_\omega\boldsymbol{\Lambda}_\omega\mathbf{Z}_\omega^{\mathbf{T}}\boldsymbol{\omega}$$

subject to

- $\forall\, t = 1, \cdots, T, j = 1, \cdots, J,$ and $f = 1, \cdots, F$:

$$\frac{\mathcal{M}_t}{N}\sum_{i=1}^{N}\pi_{ijt} = q_{jt}; \qquad \widehat{p}_{jt} = p_{jt}^{obs} - mc_{jt}$$

- $\forall\, t = 1, \cdots, T;\ i = 1, \cdots, N;$ and $j = 1, \cdots, J$:

  complementarities in the Nash-Bertrand Game

- $0 \leq mc_{jt} \leq p_{jt}^{obs}$
- $\beta_{ik} = \bar{\beta}_k + \sigma_{\beta k}\eta_{ik} \quad \forall\, k = 1, \cdots, K,$
- $\alpha_i = \exp(\bar{\alpha} w_i)$

and

- $mc_{jt} = \mathbf{y}_{\mathbf{jt}}^{\mathbf{T}}\boldsymbol{\phi} + \omega_{jt}.$

Appendix: The Nash-Bertrand game

$$0 \leq v_{ijt} \quad \perp \quad \frac{\mathcal{M}_t}{N}\pi_{ijt} - \sum_{\ell=1}^{J} \lambda_{ij\ell t} \geq 0,$$

$$\forall\, i = 1, \cdots, N; j = 1, \cdots, J; t = 1, \cdots, T$$

$$0 \leq \widehat{p}_{jt} \quad \perp \quad -\sum_{i=1}^{N} \sum_{j' \in J_f} \lambda_{ij'jt} \geq 0,$$

$$\forall\, j = 1, \cdots, J; t = 1, \cdots, T$$

$$0 \leq \lambda_{ij\ell t} \quad \perp \quad v_{ij\ell t} + \widehat{p}_{\ell t} - \frac{\mathbf{x_{\ell t}^T}\boldsymbol{\beta_i} - \alpha_i mc_{jt} + \xi_{\ell t}}{\alpha_i} \geq 0,$$

$$\forall\, i = 1, \cdots, N; j = 1, \cdots, J; \ell = 1, \cdots, J; t = 1, \cdots, T$$

$$0 \leq \pi_{ijt} \quad \perp \quad \gamma_{it} + \alpha_i \widehat{p}_{jt} - (\mathbf{x_{jt}^T}\boldsymbol{\beta_i} - \alpha_i mc_{jt} + \xi_{jt}) \geq 0,$$

$$\forall\, i = 1, \cdots, N; j = 1, \cdots, J; t = 1, \cdots, T$$

$$0 \leq \gamma_{it} \quad \perp \quad 1 - \sum_{j=1}^{J} \pi_{ijt} \geq 0.$$

$$\forall\, i = 1, \cdots, N; t = 1, \cdots, T$$

MPEC as an extension of NLP

Problematic! The existence of Lagrange multipliers is not guaranteed.

Resolution: MPEC stationary conditions and MPEC constraint qualification

◦ We will use the MPEC formulation (on p.2) but eliminate the side equality constraint for the following definitions.

MPEC active set

Definition: For a feasible point z, the *MPCC-active set* is given by the active constraint indices

$$I_g(z) = \{i : g_i(z) = 0\}$$

$$I_r(z) = \{i : r_i(z) = 0\}$$

$$I_s(z) = \{i : s_i(z) = 0\}$$

MPEC stationarity

Definition: Let z be feasible for MPEC. We say z is *B-stationary or primal stationary* if for each partition $I \cup J$ of $\{1, \ldots, m\}$ such that $I \supset I_r(z)$ and $J \supset I_s(z)$, z is stationary for $(NLP(I))$:

$$\begin{aligned} \min_z \quad & f(z) \\ \text{s.t.} \quad & g(z) \geq 0 \\ & r_I(z) = 0 \leq s_I(z) \\ & r_J(z) \geq 0 = s_J(z) \end{aligned}$$

Note: Other MPEC-stationarity includes strong-stationarity, weak-stationarity, C-stationary and A-stationarity.

MPEC-LICQ

Definition: Let z be feasible for the MPEC. The *MPEC-LICQ* holds at z if the MPEC-active constraint gradients

$$\{\nabla_z g_i(z) : i \in I_g(z)\} \cup \{\nabla_x r_i(z) : i \in I_r(z)\} \cup \{\nabla_z s_i(z) : i \in I_s(z)\}$$

are linearly independent.

Equivalent NLP

$$
\begin{array}{lrcl}
\min\limits_{z} & f(z) & & \\
\text{s.t.} & g(z) & \geq & 0 \\
 & r(z), s(z) & \geq & 0 \\
 & r(z)^T s(z) & \leq & 0.
\end{array}
$$

Proposition: Let z^* be feasible for the MPEC at which MPEC-LICQ holds. If z^* is a local minimum of the equivalent NLP, then z^* is a local minimum of the MPEC, z^* is a stationary point of the equivalent NLP, and the KKT multipliers exist for the equivalent NLP.

Note: Similar results can be obtained for formulation of $r(z)^T s(z) = 0$, $r_i(z)s_i(z) = 0, \forall i$, and $r_i(z)s_i(z) \leq 0, \forall i$.

Algorithms for solving MPEC

Methods extended from NLP:

◦ `SQP-Filter` code (Fletcher and Leyffer)

Methods for solving CP:

◦ `PATH` solver (Dirkse, Ferris, and Munson): a generalization of Newton's method
◦ Lemke's method: tableau pivotal

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