On operations research topics in the competitive environment

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Outline

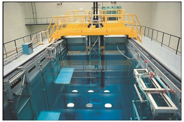
Education and Affiliation

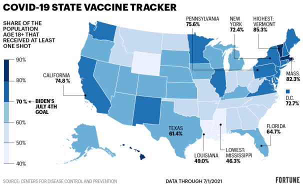
Assistant Professor, National Tsing Hua University, Hsinchu
- Equilibrium and Data-analytics Laboratory

Ph.D. Industrial Engineering 2012
University of Illinois at Urbana-Champaign
- global optimization, complementarity problems

M.S. Civil Engineering
University of Illinois at Urbana-Champaign
- transportation, airline problems

B.S. Civil Engineering
National Taiwan University





Topic 1:
Frequency Competition Among Airlines on Coordinated Airports Network

Joint work with
Chun-Han Wang (National Tsing Hua University),
Wenzhu Zhang (National Tsing Hua University),
Yue Dai (Fudan University)

Introduction::Frequency Competition

- Flight frequency has been continually increasing
- Two major tools of competition: price competition and frequency competition.
- They are very different in nature.
- Low-cost carriers compete only on price. Full-service carriers compete both on price and frequency.
- Higher flight frequency results in a greater market share. (S-Curve model often describes the relationship between flight frequency and market share.)

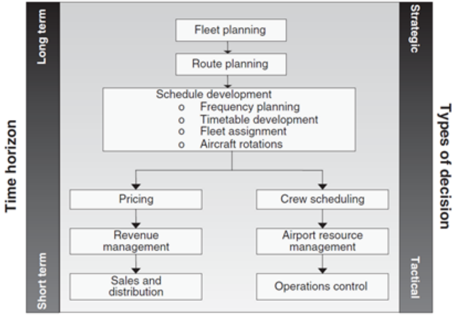
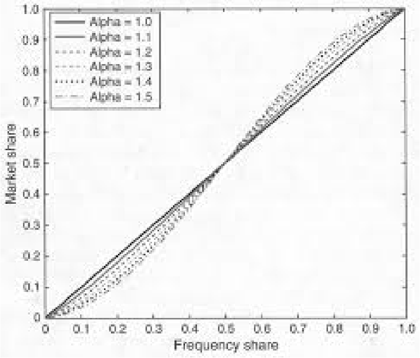


Figure 7.1 The airline planning process. (*Source*: Professor Cynthia Barnhart)

(Source: Barnhart, C., Smith, B. (2012). Quantitative problem solving methods in the airline industry. Heidelberg: Springer.)

Introduction::Frequency Competition (conti'd)

- Full-service airlines are motivated to offer dense networks of frequent services to acquire more passengers and, thus, increase their profits.
- Excessive frequency may cause diseconomies of scale and airlines' profits are restricted by their limited seats provision.
- Needing optimal decisions to fine tune frequency in order to better match their demand and supply.



(Source: Vaze, V., Barnhart, C. (2012). Modeling airline frequency competition for airport congestion mitigation. Transportation Science, 46, 512–53

Introduction::Time Slots and Slot-Constrained Airports

- **Time Slot:** "a permission granted by the owner of an airport designated as level 3, which allows the grantee to schedule a landing or departure at that airport during a specific time period".
- **Slot allocation:** according to the guidelines issued by International Air Transport Association (IATA) Worldwide Airport Slots Group
  - "Grandfather's right"
  - "Use it or lose it" rule
- Airports are categorized according to congestion level into level 1 (non-coordinated airport), level 2 (schedule-facilitated airport), and level 3 (coordinated airport)
- There are close to 200 level 3 airports out of 3,800 airports in the world.
  - In **the U.S.**: Ronald Reagan Washington National Airport **(DCA)**, John F. Kennedy International Airport **(JFK)**, and LaGuardia Airport **(LGA)**
  - The density of level 3 airports in the U.S. is lower than that in other continents, where the lack of infrastructure is sometimes hardly resolved.

Introduction::Time Slots and Slot-Constrained Airports (conti'd)

- Although, compared to the total number, level 3 airports account for only a small portion, 43% of global traffic departs from level 3 airports (IATA, 2020)
- A delay occurs at the level 3 airport will propagate in the subsequent schedule.
- The percentage of international flights involving at least one level 3 airports is higher than that of domestic flights. Nevertheless, passengers are more sensitive to the inter-flight time when taking short-haul flights.
- Frequency adjustment will have a greater effect on passengers taking short-haul domestic flights than those taking long-haul international flights.
- **The modeling logic:** Frequency should be planned for any airports and segments. Furthermore, when planning for flights departing and arriving at congested (level 3) airports, the competitive environment should be addressed.

Introduction::Airline Alliances

- **Antitrust Immunity:** The U.S. Department of Transportation's (DOT's) policy of "granting immunity from U.S. antitrust laws (ATI) for coordination on schedules and fares" by members of the three large international airline alliances: Star Alliance, oneworld, and SkyTeam.
- Some argue that it is beneficial to form an alliance because it contribute to streamlining costs and increasing competitiveness, better coordinating supply and demand, raising barriers to new entrants, reshaping industry structure, and overcoming government restrictions (Bissessur Alamdari,1998; Goh Uncles, 2003).
- By contrast, some believe that alliance formation intensifies competition both between and within alliances and that small airlines are treated unfairly in an alliance (Suen, 2005).
- Therefore, it is crucial to investigate the impact that an alliance has on frequency competition.

Literature Review

- Following Hansen (1990), Dobson & Lederer (1993) and Vaze & Barnhart (2012), we propose the game-theoretic model to solve the frequency competition problem.
  - We consider a multi-agent problem and maximize airline profits in a network system with multiple airports and include flow balance conditions in slot-constrained airports.
- Vaze & Barnhart (2012) mainly compute a pure-strategy Nash equilibrium using the best-response algorithm, which can compute an exact equilibrium when there are two players and an approximate equilibrium when there are more than two players.
  - We show that a pure-strategy Nash equilibrium may not always exist and derive the necessary conditions for this statement. Then, we formulate the problem with a mixed-strategy Nash equilibrium programming model since every finite game has at least one mixed-strategy Nash equilibrium.

Literature Review (conti'd)

- Best-response theory has been applied jointly with different numerical techniques to solve or estimate the Nash equilibrium (Wei & Hansen, 2007; Aguirregabiria & Ho, 2012; Vaze & Barnhart, 2012)
  - We apply the KKT systems, which are based on Lagrange multipliers. The technique is commonly used to solve the generalized Nash equilibrium problem (GNEP) by concatenating all KKT conditions corresponding to each player's optimization problem.

Notation

Symbol	Description
A	Set of airlines
K	Set of alliances
U^k	Set of airlines in alliance k, $k \in K$
N^a/N^k	Set of feasible strategies played by airline a/alliance k
$\mathbb{G}_{\mathbb{A}}/\mathbb{G}_{\mathbb{K}}$	Set of feasible strategy profiles played simultaneously by all airlines/alliances
I	Set of origin airports (level 3 and congested airports), e.g., DCA, LGA and JFK
J	Set of destination airports
P_{aij}	Average fare charged by airline a from airport i to j
C_{aij}	Operating cost per flight for airline a from airport i to j
S_{aij}	Seating capacity of each flight of airline a from airport i to j
U_{ai}	Maximum number of flights that can be scheduled by airline a at airport i
L_{ai}	Minimum number of flights that must be scheduled by airline a at airport i
M_{ij}	Total passenger demand from airport i to j
T_{ij}	Total flights operated from airport i to j
UT_{ij}	Upper bound of total flights operated from airport i to j
LF_{ij}	Aircraft maximal allowable load factor from airport i to j
ϵ	Tolerance level of the flow imbalance in slot-constrained airports
f_a/f_k	Strategy of service frequencies for airline a/alliance k
Q_{aij}	Number of passengers carried by airline a from airport i to j
f_{aij}	Number of flights operated by airline a from airport i to j

Table: Key Notation

Airline's optimization problem — Model I

$$\underset{f_{aij}, Q_{aij}}{\text{maximize}} \sum_{i \in I} \sum_{j \in J} (P_{aij} Q_{aij} - C_{aij} f_{aij}) \tag{1}$$

$$\text{subject to } Q_{aij} \leq \frac{f_{aij}}{\sum_{a' \in A} f_{a'ij}} M_{ij}, \quad \forall i \in I, j \in J, \tag{2}$$

$$Q_{aij} \leq LF_{ij} S_{aij} f_{aij}, \quad \forall i \in I, j \in J, \tag{3}$$

$$\sum_{j \in J} f_{aij} \leq U_{ai}, \quad \forall i \in I, \tag{4}$$

$$\sum_{j \in J} f_{aij} \geq L_{ai}, \quad \forall i \in I, \tag{5}$$

$$\left| \sum_{j \in I \setminus \{d\}} f_{adj} - \sum_{i \in I \setminus \{d\}} f_{aid} \right| \leq \epsilon, \quad \forall d \in I, \tag{6}$$

$$f_{aij}, Q_{aij} \in \mathbb{Z}_0^+, \quad \forall i \in I, j \in J. \tag{7}$$

— Constraints (2) and (3) state that the number of passengers for airline a from airport i to j is restricted by both demand (i.e., market share) and supply (i.e., available seats).

— Constraints (4) and (5), comprising the upper and lower bounds of the number of flights for airline a, denote that the utilization of take-off slots should reach the minimum level but not exceed the maximum level.

— Constraint (6) requires that the difference between the numbers of flights arriving at and departing from airport i should not exceed the tolerance level ϵ.

Assumption and Result

Assumption: $P_{aij}\lfloor LF_{ij}S_{aij}f_{aij}\rfloor - C_{aij}f_{aij} > 0$, $\forall a \in A$, $i \in I$, $j \in J$, $M_{ij}LF_{ij}S_{aij} \neq 0$ and $f_{aij} \in \mathbb{N}$.

Theorem

If there exists pure-strategy Nash equilibrium $(f^*_{a_1}, f^*_{\tilde{a}_1})$, then $\forall a \in A$, $i \in I$, statements (i) and (ii) hold:

(i) One of the two conditions (a1) and (a2) is true.

(ii) One of the two conditions (b1) and (b2) is true.

Conditions (a1), (a2), (b1) and (b2) are defined as follows:

(a1) $\exists\ j \in J$, and $M_{ij}LF_{ij}S_{aij} \neq 0$ such that $\dfrac{M_{ij}}{LF_{ij}S_{aij}} \geq \sum_{a' \in A} f^*_{a'ij} + 1$ and $\sum_{j' \in J} f^*_{aij} = U_{ai}$.

(a2) $\dfrac{M_{ij}}{LF_{ij}S_{aij}} < \sum_{a' \in A} f^*_{a'ij} + 1$, $\forall j \in J$, $M_{ij}LF_{ij}S_{aij} \neq 0$.

Assumption and Result (conti'd)

Theorem (conti'd)

(b1) $\exists\, j \in J$, and $M_{ij} LF_{ij} S_{aij} \neq 0$ such that $\dfrac{P_{aij} M_{ij}}{C_{aij} - P_{aij}} < f^*_{aij} - 1$ and $\displaystyle\sum_{j' \in J} f^*_{aij} = L_{ai}$.

(b2) $\dfrac{P_{aij} M_{ij}}{C_{aij} - P_{aij}} \geq f^*_{aij} - 1, \forall j \in J,\ M_{ij} LF_{ij} S_{aij} \neq 0$.

We proved by contradiction.

- This is a necessary condition for the existence of a pure-strategy Nash equilibrium.

Assumption and Result (conti'd)

Corollary

If there exist $a \in A$ and $i \in I$ such that

$$\sum_{j \in J} \frac{M_{ij}}{LF_{ij} S_{aij}} \geq \sum_{a' \in A} U_{a'i} + \|J\|$$

and

$$\sum_{j \in J} \frac{P_{ij} M_{ij}}{C_{aij} - P_{aij}} < L_{ai} - \|J\|,$$

then Nash equilibrium must not exist when $L_{ai} < U_{ai}$.

- Corollary implies that we may not find a pure-strategy Nash equilibrium.
- As done in this study, all parameters can be estimated with the data from the BTS website to determine whether a pure-strategy Nash equilibrium can exist.

Mixed-Strategy Game Formulation

- Since a pure-strategy Nash equilibrium may not always exist, we reformulate the problem using a mixed-strategy model.
- A mixed strategy is an assignment of probability to each pure strategy. In a simultaneous action game, the players decide their action simultaneously without knowledge of each other's decision.
- We consider that each airline will expect other airlines' total number of flights to be the previous operations, and compute a reasonable strategy according to the number. Airlines will simultaneously determine their own decisions based on these strategies.

Symbol	Description
$\pi_a(f_a, f_{\tilde{a}})$	Profits of airline a with strategy profile $(f_a, f_{\tilde{a}})$, $a \in A, \tilde{a} \in A\backslash\{a\}$
$\pi_k\left(f_k, f_{\tilde{k}}\right)$	Profits of alliance k with strategy profile $\left(f_k, f_{\tilde{k}}\right)$, $k \in K, \tilde{k} \in K\backslash\{k\}$
$g_a(f_a)$	Probability that airline a adopts strategy f_a
$g_k(f_k)$	Probability that alliance k adopts strategy f_k

Table: More Notation

Airline's optimization problem with mixed strategy—Model II

$$\underset{g_a(f_a)}{\text{maximize}} \sum_{(f_a, f_{\tilde{a}}) \in \mathbb{G}_{\mathbb{A}}} \pi_a(f_a, f_{\tilde{a}}) \cdot g_a(f_a) \cdot \prod_{\tilde{a} \in A \setminus \{a\}} g_{\tilde{a}}(f_{\tilde{a}}) \tag{8}$$

$$\text{subject to} \sum_{f_a \in N^a} g_a(f_a) = 1, \tag{9}$$

$$g_a(f_a) \geq 0, \quad \forall f_a \in N^a. \tag{10}$$

Constraint (9) implies that the sum of probability that airline a selects over the full set of strategies equals one and constraint (10) states that the probability should be non-negative.

Mixed-strategy Nash equilibrium must exists. There might be multiple MSNE. To compute one, we solve the following KKT system of all airlines

$$\sum_{(f_a, f_{\tilde{a}}) \in \mathbb{G}_{\mathbb{A}}} \pi_a(f_a, f_{\tilde{a}}) \cdot \prod_{\tilde{a} \in A \setminus \{a\}} g_{\tilde{a}}(f_{\tilde{a}}) + \lambda_a + \mu_{f_a} = 0, \quad \forall a \in A, f_a \in N^a,$$

$$\sum_{f_a \in N^a} g_a(f_a) = 1, \quad \forall a \in A,$$

$$g_a(f_a) \geq 0, \quad \forall a \in A, f_a \in N^a, \mu_{f_a} \cdot g_a(f_a) = 0, \quad \forall a \in A, f_a \in N^a, \mu_{f_a} \geq 0, \quad \forall f_a \in N^a,$$

Extend the model of the Airline Scenario to the Alliance Scenario

In the objective function, we consider the profits of alliance k.

$$\underset{g_k(f_k)}{\text{maximize}} \sum_{(f_k, f_{\tilde{k}}) \in \mathbb{G}_{\mathbb{K}}} \pi_k(f_k, f_{\tilde{k}}) \cdot g_k(f_k) \cdot \prod_{\tilde{k} \in K \setminus \{k\}} g_{\tilde{k}}(f_{\tilde{k}})$$

$$\text{subject to} \sum_{f_k \in N^k} g_k(f_k) = 1,$$

$$g_k(f_k) \geq 0, \quad \forall f_k \in N^k,$$

where the feasible strategy has the centralized seat provision mechanism

$$Q_{aij} \leq \frac{f_{aij}}{\sum_{a \in A} f_{aij}} M_{ij}, \quad \forall a \in U^k, i \in I, j \in J,$$

$$\sum_{a \in U^k} Q_{aij} \leq \sum_{a \in U^k} LF_{ij} S_{aij} f_{aij}, \quad \forall i \in I, j \in J,$$

$$\sum_{j \in J} f_{aij} \leq U_{ai}, \quad \forall a \in U^k, i \in I,$$

$$\sum_{j \in J} f_{aij} \geq L_{ai}, \quad \forall a \in U^k, i \in I,$$

$$\left| \sum_{j \in I \setminus \{d\}} f_{adj} - \sum_{i \in I \setminus \{d\}} f_{aid} \right| \leq \epsilon, \quad \forall a \in U^k, d \in I,$$

$$f_{aij}, Q_{aij} \in \mathbb{Z}_0^+, \quad \forall a \in U^k, i \in I, j \in J.$$

Empirical Analysis::Input Parameters Estimation

- All the data is downloaded from the Bureau of Transportation (BTS) website in June, 2016.
- Although our proposed equilibrium programming approach does not specify borders limit and can be adopted for frequency analysis on domestic and international airports networks, data of the U.S. airlines market is relatively open.
- We focus on the departure of domestic flights from the three airports and on five airlines which operate at least one of the 71, 92 and 77 segments departing from JFK, DCA and LGA, respectively.
- The five airlines are selected from three different alliances: SkyTeam (ST), Oneworld (OW), and Star Alliance (SA).
- The historical data for the airlines are used to conduct a model-based empirical analysis.

Empirical Analysis::Feasible Strategy Generation

- Enumerating strategies set of full size is unrealistic.
- For five airlines and three alliances, one needs to store 5^5 and 9^3 entries of profits π_a and π_k, respectively, which has reached our computer memory limits.
- We let the total number of flights from airport i to j (i.e., $\sum_{a \in A} f_{aij}$) in constraint (2) be a constant T_{ij}, which is the total frequency from airport i to j in June 2016 according to BTS. We next add the objective to maximize the industry's total profits. By solving the revised maximization model, we obtain the first strategy referred to as the **centralized strategy**.
- We select competitive segments, i.e., segments operated by more than two airlines. Perturb the centralized strategy randomly and generate the remaining strategies.

Empirical Analysis::Mixed Strategies (Airline Scenario)

Airline	First strategy	Second strategy	Third strategy	Fourth strategy	Fifth strategy	Expected Profits	Expected Pax	Expected Profits per Pax
			($MSNE_{max}$)					
A	0.9479	0.0166	0.0166	0.0103	0.0084	30,020,000	176,941	169.6606
B	0.0017	0.0060	0.0188	0.0136	0.9597	108,755,000	696,093	156.2362
C	0.0143	0.0120	0.9489	0.0120	0.0126	141,392,000	708,078	199.6841
D	0.9702	0.0111	0.0055	0.0065	0.0065	21,345,100	111,487	191.4579
E	0.0932	0.4410	0.3140	0.1161	0.0354	10,862,700	130,419	83.2906
			($MSNE_{median}$)					
A	0.9892	0.0033	0.0033	0.0022	0.0018	30,037,900	177,046	169.6617
B	0.0039	0.0012	0.0037	0.0026	0.9884	108,730,000	696,006	156.2200
C	0.0031	0.0029	0.9879	0.0029	0.0029	141,376,000	707,988	199.6869
D	0.9924	0.0028	0.0014	0.0016	0.0016	21,353,000	111,532	191.4526
E	0.0152	0.8253	0.1334	0.0205	0.0053	10,877,500	131,314	82.8358
			($MSNE_{min}$)					
A	0.9474	0.0167	0.0167	0.0104	0.0085	30,019,200	176,938	169.6597
B	0.0068	0.0056	0.0190	0.0131	0.9552	108,758,000	696,106	156.2378
C	0.0208	0.0153	0.9335	0.0153	0.0148	141,389,000	708,081	199.6791
D	0.9645	0.0137	0.0064	0.0076	0.0076	21,342,600	111,474	191.4572
E	0.1049	0.4006	0.3281	0.1322	0.0340	10,861,600	130,334	83.3366

Table: Probability distribution for each airline's frequency strategy at three mixed-strategy Nash equilibria

Empirical Analysis::Mixed Strategies (Alliance Scenario)

Alliance	First strategy	Second strategy	Third strategy	Fourth strategy	Fifth strategy	Sixth strategy	Seventh strategy	Eighth strategy	Ninth strategy
					($MSNE_{max}$)				
ST	0.8426	0.0000	0.0000	0.0000	0.0000	0.0384	0.0279	0.0277	0.0631
OW	0.8995	0.0000	0.0000	0.0000	0.0000	0.0250	0.0253	0.0248	0.0251
SA	0.0941	0.0941	0.0941	0.0941	0.0436	0.1411	0.1428	0.1478	0.1478
					($MSNE_{median}$)				
ST	0.5555	0.0000	0.0000	0.0000	0.0000	0.0875	0.1228	0.0980	0.1359
OW	0.5941	0.0000	0.0000	0.0000	0.0000	0.1068	0.1279	0.0640	0.1069
SA	0.0000	0.0000	0.0000	0.0000	0.0000	0.2384	0.2444	0.2585	0.2585
					($MSNE_{min}$)				
ST	0.0000	0.0000	0.0000	0.0000	0.0000	0.2492	0.2494	0.2505	0.2507
OW	0.0000	0.0000	0.0000	0.0000	0.0000	0.0729	0.3130	0.3126	0.3013
SA	0.0475	0.4706	0.0475	0.0475	0.0475	0.0571	0.0580	0.0572	0.1667

Table: Probability distribution of each alliance's frequency strategy at three mixed-strategy Nash equilibria

Empirical Analysis::Profits Comparisons

- According to Figure, an airline may not earn more profits under the mixed strategy from the alliance scenario than those from the airline scenario.
- However, total profits earned by airlines of an alliance under the mixed strategy from the alliance scenario are higher than those from the airline scenario.
- Through a reasonable revenues allocation, the increase in profitability gives airlines an incentive to form an alliance.

	Status Quo	Centralized Strategy Under Airline Scenario	Airline Scenario	Centralized Strategy Under Alliance Scenario	Alliance Scenario
Min			312,370,400		316,916,700
Mid			312,374,400		319,256,600
Max	289,537,200	312,213,900	312,374,800	320,569,300	320,198,800

Figure: Total industry's profits

Empirical Analysis::Number of Passengers Comparisons

- According to Figure, we find that comparing to status quo, airlines will serve more passengers under both airline scenario and alliance scenario, which is true for min, median and max cases.
- That is, with our algorithm, more passengers will be served.
- What's more, forming alliance enables the whole industry to serve more passengers.

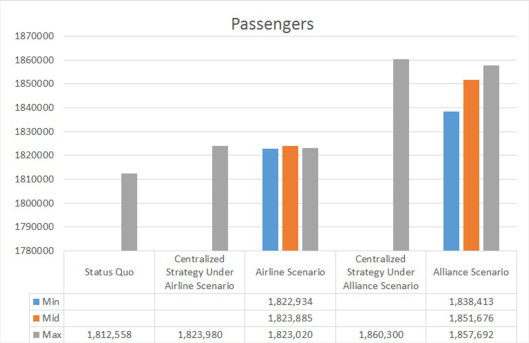


Figure: Total number of passengers

Conclusions

- Forming a pure strategy profile in frequency competition among airlines may naturally lead to **deviation from current frequency.**
- Our empirical results indicate that adopting a mixed strategy can increase total industry's profitability by 7.89%. We extend the model to formulate frequency competition among neutral metal alliances and show that forming neutral metal alliances can improve the total industry's profits by 10.59%.
- In particular, a sensitivity analysis on the tolerance level of flow imbalance shows that deducting the potential costs due to the set of tolerance between congested airports may generate 0.36% additional profits in frequency competition with real data.
- Although the lower limits of time slots usage is currently set according to the 80% rule, the model we proposed has flexibility to set a smaller lower limit. In future study, more **complicated slots transactions** between airlines including **trading and leasing** can be taken into account in frequency competition.

Topic 2: Competitive Demand Learning:
A Coordinated Price Experimentation
and Non-cooperative Pricing Algorithm

Joint work with
Yongge Yang (National Tsing Hua University),
Po-An Chen (National Yang Ming Chiao Tung University).

Introduction about this work

- Consider a total of F firms selling substitutable products in an oligopolistic market, in which the true underlying demand curve and the presence of demand shocks are unknown. Over a time horizon of T periods, firms make pricing decisions in each period $t = 1, \cdots, T$.
- The price decisions made by other firms will influence the demand for the product of firm i.
- By focusing on competition among the F firms, we do not consider capacity limitation, production cost or marginal cost. Each firm is assumed to be selfish and reacts immediately to price changes made by competitors.

Introduction about this work (conti'd)

- This paper generalizes the work of Besbes and Zeevi (2015), who constructed a dynamic pricing algorithm in a monopoly setting in which a single firm chooses a price to maximize expected revenue without knowledge of the true underlying demand curve.
- We propose an equilibrium pricing algorithm to solve the dynamic pricing decisions of each firm in competition.
- In an alternative scenario, in which some firms have knowledge of the demand function and the distribution of demand shocks, such firms may be unwilling to engage in price experimentation. Therefore, we propose a modified DDEP algorithm (in the full paper) to account for this.
- The process of learning is often evaluated in terms of *regrets*.
- We also analyze the revenue difference obtained by the algorithm from that obtained by the clairvoyant Nash equilibrium p^* per algorithm iteration.

Related Literature

- **Learning algorithms for pricing models in a monopoly market.**
  - Besbes O, Zeevi A (2015) On the (surprising) sufficiency of linear models for dynamic pricing with demand learning. Management Science 61(4):723–739.
  - Chen B, Chao X, Ahn HS (2019) Coordinating pricing and inventory replenishment with nonparametric demand learning. Operations Research 67(4):1035–1052.
  - Chen B, Chao X, Shi C (2015) Nonparametric algorithms for joint pricing and inventory control with lost sales and censored demand. Mathematics of Operations Research, Major Revision.
- **Dynamic pricing in a competitive environment.**
  - Cooper WL, Homem-de Mello T, Kleywegt AJ (2015) Learning and pricing with models that do not explicitly incorporate competition. Operations research 63(1):86–103.
- **Learning algorithms for pricing models in a competitive environment.**
  - Bertsimas D, Perakis G (2006) Dynamic pricing: A learning approach. Mathematical and computational models for congestion charging, 45–79 (Springer)
  - Gallego G, Talebian M (2012) Demand learning and dynamic pricing for multi-version products. Journal of Revenue and Pricing Management 11(3):303–318.
  - Fisher M, Gallino S, Li J (2018) Competition-based dynamic pricing in online retailing: A methodology validated with field experiments. Management Science 64(6):2496–2514.

Model and Preliminaries

We consider a periodical equilibrium pricing problem for F firms. In each period $t = 1, \cdots, T$, each firm needs to set prices p_t^i, chosen from a feasible and bounded policy set $\mathcal{P}^i = \left[p^{i,\ell}, p^{i,h}\right]$, $p^{i,\ell} < p^{i,h}$, $\forall i = 1, \cdots, F$. The prices set by firms affect the market response of all firms in the competition.

Recall that $\mathrm{p} \equiv (p^i, p^{-i})$ denotes the vector of prices of all firms in the competition. The market response to the price p_t^i for firm i at time t (which is exactly the demand function) is given by $D_t^i(\mathrm{p}_t) = \lambda^i(\mathrm{p}_t) + \varepsilon_t^i$, $\forall i = 1, \cdots, N$, in which $\lambda^i(\mathrm{p}_t)$ is a deterministic twice differentiable function representing the mean demand, conditional on the price p_t

ε_t^i are zero-mean random variables, assumed to be independent and identically distributed.

Model and Preliminaries (conti'd)

Hence, the demand curve $\lambda^i(\mathrm{p})$ of firm i not only depends on the price p^i, chosen by itself, but also on the prices of other firms p^{-i}, where $p^{-i} = \{p^1, \cdots, p^{i-1}, p^{i+1}, \cdots, p^F\}$.

Let $\pi^i = (p^i_1, p^i_2, \cdots)$ denote the sequence of pricing policy of firm i and $\Pi = (\pi^1, \cdots, \pi^F)$ denote the admissible pricing policies of all firms.

The revenue function r^i of firm i obtained from prices p is denoted by $r^i(\mathrm{p}) \equiv p^i \mathbb{E}[D^i(\mathrm{p})]$. Each firm seeks to maximize its revenue in a competitive environment.

Model and Preliminaries (conti'd)

Throughout the paper, we use period t or, equivalently, time t.

Let p_t^{i*} denote the equilibrium price of firm i at time t, which is obtained by the estimated demand curve of firm i at time t, and let p_t^{-i} denote the prices of other competitors.

A *clairvoyant* model implies that a firm has knowledge of the underlying demand curve and the distribution of demand shocks.

The goal of learning is to make p_t^i converge to the clairvoyant equilibrium price of firm i, p^{i*}, as t grows large. Note that a learning scheme in which the difference between p_t^i and p^{i*} will eventually converge to zero is called *complete* learning; otherwise, it is termed *incomplete* learning.

Assumptions

(i) For any $p^i \in \mathcal{P}^i$, $\dfrac{\partial \lambda^i(\cdot, p^{-i})}{\partial p^i} < 0, \forall i = 1, \cdots, F$.

(ii) For any $p^i \in \mathcal{P}^i$, $\dfrac{\partial \lambda^i(p^i, p^{-i \setminus j}, \cdot)}{\partial p^j} > 0, \forall j \neq i, i = 1, \cdots, F$.

(iii) For any $p^i \in \mathcal{P}^i$, $\dfrac{\partial^2 r^i(\mathrm{p})}{\partial^2 \mathrm{p}}, \forall i = 1, \cdots, F$ is a negative definite matrix.

(iv) For every $r^i(\mathrm{p})$,

$$\sum_{j \neq i}^{F} \left| \frac{\partial^2 r^i}{\partial p^i \partial p^j} \right| < \left| \frac{\partial^2 r^i}{\partial p^{i2}} \right|, \forall p^i \in \mathcal{P}^i, p^j \in \mathcal{P}^j.$$

(v) For every firm i, there exists a constant s_0 such that, for all $|s| < s_0$, $\mathbb{E}\left[\exp\left\{s\varepsilon_1^i\right\}\right] < \infty$, and the variance of each firm's ε^i is equal to σ^2.
(vi) For every firm i, given p^{-i}, firm i chooses to price at a $p^i \in \mathcal{P}^i$ satisfying $\mathbb{E}\left[D^i(p^i, p^{-i})\right] \geq 0$.

Competitive Demand Learning (CDL) algorithm

loop n from 0 until a terminal stage, given as period T.

- Step 0. Preparation: If $n = 0$, input I_0, v, and $\hat{p}_1^i, \forall i = 1, \cdots, F$. If $n > 0$, set $I_n = \lfloor I_0 v^n \rfloor$ and $\delta_n = I_n^{-\frac{1}{4}}$.
- Step 1: Setting prices. Loop m from 1 to $F+1$. The rule of firm i's price p_t^i at time t is

$$\begin{aligned} &\text{if } m \neq i, \\ &p_t^i = \hat{p}_n^i, \qquad \forall t = t_n + 1, \cdots, t_n + iI_n, t_n + (i+1)I_n + 1, \cdots, t_n + (F+1)I_n, \\ &\text{if } m = i, \\ &p_t^i = \hat{p}_n^i + \delta_n, \quad \forall t = t_n + iI_n + 1, \cdots, t_n + (i+1)I_n. \end{aligned}$$

  End the m-loop. Set $t_{n+1} = t_n + (F+1)I_n$.
- Step 2. Estimating:

$$(\hat{\alpha}_{n+1}^i, \hat{\beta}_{n+1}^{ij}) = \arg\min_{\alpha^i, \beta^{ij}} \left\{ \sum_{t=t_n+1}^{t=t_n+(F+1)I_n} \left[D_t^i - \left(\alpha^i - \beta^{ii} p_t^i + \sum_{j=1, j\neq i}^{F} \beta^{ij} p_t^j \right) \right]^2 \right\}.$$

- Step 3. Computing the equilibrium: We define the following optimization problem for firm i: $\max_{p^i} r^i_{n+1} \equiv \max_{p^i} G_{n+1}\left\{p^i, p^{-i}, \hat{\alpha}^i_{n+1}, \hat{\beta}^{ij}_{n+1}\right\}$, where $G_{n+1}\left\{p^i, p^{-i}, \hat{\alpha}^i_{n+1}, \hat{\beta}^i_{n+1}\right\}$ $\equiv \left\{ p^i \left(\hat{\alpha}^i_{n+1} - \hat{\beta}^{ii}_{n+1} p^i + \sum_{j=1, j\neq i}^{F} \hat{\beta}^{ij}_{n+1} p^j \right) \middle| \hat{\alpha}^i_{n+1} - \hat{\beta}^{ii}_{n+1} p^i + \sum_{j=1, j\neq i}^{F} \hat{\beta}^{ij}_{n+1} p^j \geq 0, p^i \in \mathcal{P}^i \right\}$. Proceeding to solve the system:

$$\left[\ \hat{\alpha}^i_{n+1} - 2\hat{\beta}^{ii}_{n+1} p^i + \sum_{j, j\neq i}^{F} \hat{\beta}^{ij}_{n+1} p^j \ \right] + \mu^{i,1}\left[\ -\hat{\beta}^i_{n+1}\ \right] - \mu^{i,2} + \mu^{i,3} = 0 \qquad \forall i,$$

$$\mu^{i,1} \geq 0, \mu^{i,1} \cdot \left(-\hat{\alpha}^i_{n+1} + \hat{\beta}^{ii}_{n+1} p^i - \sum_{j, j\neq i}^{F} \hat{\beta}^{ij}_{n+1} p^j \right) = 0, \hat{\alpha}^i_{n+1} - \hat{\beta}^{ii}_{n+1} p^i + \sum_{j, j\neq i}^{F} \hat{\beta}^{ij}_{n+1} p^j \geq 0 \qquad \forall i,$$

$$\mu^{i,2} \geq 0, \mu^{i,2} \cdot \left(p^i - p^{i,h}\right) = 0, p^{i,h} - p^i \geq 0 \qquad \forall i,$$

$$\mu^{i,3} \geq 0, \mu^{i,3} \cdot \left(p^{i,l} - p^i\right) = 0, p^i - p^{i,l} \geq 0 \qquad \forall i.$$

Then, prices for each firm $\hat{p}^i_{n+1}$ are set to the unique solution of this system. Set $n = n + 1$ and **return to Step 0**.

Assumptions Implications

(i) ensures that for every firm i, the underlying demand function $\lambda^i(\cdot, p^{-i})$ is strictly decreasing on p^i given the prices set by other firms, p^{-i}.
(ii) dictates that $\lambda^i(p^i, p^{-i\setminus j}, \cdot)$ is strictly increasing on p^j with p^i and $p^{-i\setminus j}$ given, in which $p^{-i\setminus j}$ represents the vector constituted by all prices except p^i and p^j.
(iii) dictates that the revenue function $r^i(\mathrm{p})$ is a concave function and thus there exists a unique maximizer for any feasible p.
(iv) is termed as the "diagonal dominance" condition.
(v) ensures that the demand shock ε_t^i of each firm has a light-tailed distribution and the homogeneity of variance.
(vi) ensures that each firm only considers a price within a price interval such that the estimated demand is non-negative.

Assumptions Examples

We give some examples of demand functions from that satisfy Assumption 2.(i).

1. Linear demand: $\lambda^i(\mathrm{p}) = \alpha^i - \beta^{ii} p^i + \sum_{j=1, j \neq i}^{F} \beta^{ij} p^j$, $\beta^{ii} > 0$.
2. Multinomial logit demand: $\lambda^i(\mathrm{p}) = \frac{\exp^{\alpha^i - \beta^i p^i}}{1 + \sum_{i=1}^{F} \exp^{\alpha^i - \beta^i p^i}}$, $\alpha^i > 0, \beta^i > 0$ and

$\alpha^i - \beta^i p^i < 0$ for $p^i \in \mathcal{P}^i$.

3. Cobb-Douglas demand: $\lambda^i(\mathrm{p}) = \alpha^i (p^i)^{-\beta^{ii}} \prod_{j \neq i}^{F} (p^j)^{\beta^{ij}}$, $\alpha^i > 0, \beta^{ii} > 1, \beta^{ij} \geq 0$.
4. CES demand: $\lambda^i(\mathrm{p}) = \frac{\gamma (p^i)^{r-1}}{\sum_{j=1}^{F} (p^j)^r}$, $r < 0, \gamma > 0$.

Lemma 1: Analysis for the Noiseless Case.

Lemma 1

Suppose that $\varepsilon_t^i = 0$, $\forall i$ and t, and that the sequence $\{\hat{\mathrm{p}}_n\}$, assuming nonzero demand and that the price is away from the limits, generated by CDL converges to a limit point $\tilde{\mathrm{p}}$, which satisfies $\tilde{p}^i = -\dfrac{\lambda^i(\tilde{\mathrm{p}})}{\nabla_{p^i}\lambda^i(\tilde{\mathrm{p}})}$. Then, $\tilde{\mathrm{p}}$ is exactly p^*.

Lemma 2: Uniqueness of $\hat{p}_n$.

Lemma 2

Under the assumptions, $\hat{p}_n$ is a unique GNE at stage n with high probability.

Proposition 1

Proposition 1

If $p'^i = z^i\left(\breve{\alpha}^i(p^i, p^{-i}), \breve{\beta}^{ii}(p^i, p^{-i}), \breve{\beta}^{ij}(p^i, p^{-i})\right)$ for all i, then there exists a constant $\gamma \in (0, 1)$ such that

$$\|\mathrm{p}^* - \mathrm{z}(\hat{\mathrm{p}}_n)\| \leq \gamma \|\mathrm{p}^* - \hat{\mathrm{p}}_n\|.$$

Proposition 1 is based on a deterministic (mean) demand function, and the convergence result follows directly from the property of a contraction mapping. Now, we focus on a randomized demand function and we aim to establish the convergence result as follows.

Proposition 2

Proposition 2

For any given $\hat{p}_n^i \in \mathcal{P}^i$ generated by CDL, with high probability the following inequality holds

$$\|\mathrm{z}(\hat{\mathrm{p}}_n) - \hat{\mathrm{p}}_{n+1}\| \leq \|C_n\|,$$

where $C_n \equiv \left[C_n^1, \cdots, C_n^F\right]$ is a vector of constants.

Proposition 2 shows that the difference between these best response functions is bounded with high probability. The probability lower bound is specified in the proof of Proposition 3. Note that $\mathrm{z}(\hat{\mathrm{p}}_n)$ denotes the collection of all firms' best responses $z^i\left(\breve{\alpha}^i(\hat{\mathrm{p}}_n), \breve{\beta}^{ii}(\hat{\mathrm{p}}_n), \breve{\beta}^{ij}(\hat{\mathrm{p}}_n)\right)$.

Proposition 3

Proposition 3

At stage n, for some suitable constant K_1, the operator $\mathrm{z}(\hat{\mathrm{p}}_n)$ and the CDL generated $\hat{\mathrm{p}}_{n+1}$ satisfy

$$\mathbb{E}\left[\|\mathrm{z}(\hat{\mathrm{p}}_n) - \hat{\mathrm{p}}_{n+1}\|^2\right] \leq F^2 K_1 I_n^{-\frac{1}{2}}.$$

Proposition 3 also provides an upper bound for the deviation between $\mathrm{z}(\hat{\mathrm{p}}_n)$ and $\hat{\mathrm{p}}_{n+1}$. The upper bound is related to the squared number of competitive firms, F^2, and converges to zero as the number of stages increases.

Analysis: Convergence, Revenue Difference, and Regret

Theorem 1: Convergence

Under Assumptions, the GNE, $\hat{p}_n$ converges in probability to p^* as $n \to \infty$.

The best response function derived by CDL through the quadratic concave function will generate the sequence $\{p_t\}$ that converges to p^* as t grows large. (Theorem 1)

Analysis: Convergence, Revenue Difference, and Regret

Theorem 2: Revenue Difference

Under Assumptions, the sequence of the generalized Nash equilibrium $\{\hat{\mathrm{p}}_t : t \geq 1\}$ satisfies

$$\mathbb{E}\left[\sum_{t=1}^{T}\left[\left|r^i(p^{i*}, p^{-i*}) - r^i(p_t^i, p_t^{-i})\right|\right]\right] \leq F^2 K_6 T^{\frac{1}{2}},$$

$$\forall i = 1, \cdots, F,$$

for some positive constant K_6, $T \geq 2$, and $F \geq 2$.

The revenue difference converges to zero as time progresses and is related to the quantity of firms in competition. The difference implies that realised revenues are sometimes greater than those revenues obtained by the clairvoyant Nash equilibrium p^*. However, we are unable to predict when this will occur. (Theorem 2)

Analysis: Convergence, Revenue Difference, and Regret

Theorem 3: Regret

Under the defined assumptions, the sequence of optimal decisions $\{p_t^{i*} : t \geq 1\}$ satisfies

$$\mathbb{E}\left[\sum_{t=1}^{T}\left[r^i(p_t^{i*}, p_t^{-i}) - r^i(p_t^i, p_t^{-i})\right]\right] \leq K_7 F T^{\frac{1}{2}},$$

$$\forall i = 1, \cdots, F,$$

for some positive constant K_7, $T \geq 2$ and $F \geq 2$.

As time progresses, the accumulated revenue of each firm generated by the pricing policies of CDL algorithm is asymptotically close to the clairvoyant accumulated revenue.
(Theorem 3)

Concluding Remarks

- We designed a mechanism of synchronized dynamic pricing. Such a mechanism ensures that the pricing strategy of each firm is adjusted in a prescribed way to jointly collect demand information and make pricing decisions.
- We asked whether the mechanism may allow prices to reach a stable state and how much regret firms incur by employing such a data-driven pricing algorithm.
- In particular, the effects of noise vanish as n increases and that the fitted linear model can serve as an estimation of the underlying demand model without being affected by F.
- When facing competition, the upper bound of revenue regret, derived in the same way as that of one firm, is scaled by F (i.e., Theorem 3), the upper bound of revenue difference is scaled by F^2 (i.e., Theorem 2), and the deviation between the best responses and the clairvoyant GNE price is upper bounded by a factor of $F^2 I_n^{-\frac{1}{2}}$